Boolean Algebra and binary system

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January 19, 2010
Outline

1. Boolean algebra
2. Binary systems
3. Representing information
4. Conclusions
5. Exercises
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2. Binary systems
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An algebra for logic

- Domain: \{true, false\}
- Basic operations: \{and, or, not\}
- Truth tables:

<table>
<thead>
<tr>
<th>a and b</th>
<th>false</th>
<th>true</th>
</tr>
</thead>
<tbody>
<tr>
<td>false</td>
<td>false</td>
<td>false</td>
</tr>
<tr>
<td>true</td>
<td>false</td>
<td>true</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a or b</th>
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<th>true</th>
</tr>
</thead>
<tbody>
<tr>
<td>false</td>
<td>false</td>
<td>true</td>
</tr>
<tr>
<td>true</td>
<td>true</td>
<td>true</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a</th>
<th>not a</th>
</tr>
</thead>
<tbody>
<tr>
<td>false</td>
<td>true</td>
</tr>
<tr>
<td>true</td>
<td>false</td>
</tr>
</tbody>
</table>
Examples of logic predicates

- **Axioms**
  - Today is raining
  - John carries an umbrella
  - John wears sunglasses

- **Predicates**
  - Today is raining and John carries an umbrella is true and true \( \equiv \text{true} \)
  - not today is raining or John wears sunglasses \( \equiv \) not true or true \( \equiv \text{true} \)
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Other operators

- $a \text{nand} b \equiv \neg (a \text{ and } b)$
- $a \text{ nor } b \equiv \neg (a \text{ or } b)$
- $a \rightarrow b \equiv \neg (a \text{ and } \neg b)$
- $a \text{ xor } b \equiv (a \text{ or } b) \text{ and } \neg (a \text{ and } b)$
- It can be shown that every operator can be derived by either nand or nor
Properties

- It is an algebra, thus it has the following properties:
  - the identity for and is true
  - the identity for or is false
  - the null element for and is false
  - commutativity. ex: \( a \land b \equiv b \land a \)
  - associativity. ex: \( a \lor (b \lor c) \equiv (a \lor b) \lor c \)
Boolean algebra in digital electronic systems

- It is possible to build electronic logic gates that
  - Interpret high voltage as true and low voltage as false
  - Implement logic operations like nand and nor

Figure: A logic circuit that implements $z \equiv \neg ((a \lor b) \land c)$
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Positional notation

- Humans use a positional notation in base 10
- We have 10 symbols: from 0 to 9

\[ 176435_{b(10)} = 1 \cdot 10^5 + 7 \cdot 10^4 + 6 \cdot 10^3 + 4 \cdot 10^2 + 3 \cdot 10^1 + 5 \cdot 10^0 \]

- In base 8 (octal), we have 8 symbols: from 0 to 7
- The same “number”, expressed in base 8 would be:

\[ 176435_{b(8)} = 1 \cdot 8^5 + 7 \cdot 8^4 + 6 \cdot 8^3 + 4 \cdot 8^2 + 3 \cdot 8^1 + 5 \cdot 8^0 = 64797_{b(10)} \]
In digital electronic systems, high and low voltages are interpreted as two different symbols, 1 and 0 respectively.

It is possible to build arithmetic using binary encoding of numbers and symbols.

Definitions:
- one binary digit (0 or 1) is a *bit*
- a group of 8 binary digits is a *byte*
- a *word* in current processor is 4 bytes (32 bits)
Binary encoding integer numbers

- Translation from decimal to binary and viceversa
  - Let's start from positive integer numbers
  - The minimum number is 0000 0000 (0 in decimal)
  - The maximum number is 1111 1111 (255 in decimal)
  - How to translate a binary number:

  \[ 0100 \ 1011 = \]
  \[ 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 75 \]

  \[ 0011 \ 0110 = \]
  \[ 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 1 \cdot 2^4 + 1 \cdot 2^5 + 0 \cdot 2^6 + 0 \cdot 2^7 = 54 \]
Summing integer numbers

- By using boolean logic, we can implement binary adders
- Truth table of an adder: \( s = x + y \), plus the carry

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- By interpreting 0 as `false` and 1 as `true`, the sum can be expressed as:
  - \( s = x \text{ xor } y \)
  - \( c = x \text{ and } y \)
Basic adder and full adder

The following diagram represent a 2-bit adder
Let's consider a component that can be used to build more complex adders:

- **Truth table:**

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_0$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$c_{in}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$c_{out} s_0$</td>
<td>00</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

- $s_0 = x_0 \text{xor} \ y_0 \text{xor} \ c_{in}$
- $c_{out} = (x_0 \text{ and } y_0) \text{ or } (x_0 \text{ and } c_{in}) \text{ or } (y_0 \text{ and } c_{in})$
To implement a 4-bit adder, we compose 4 full-adders:
How to represent negative numbers

There are many ways to represent negative integers

1. use the first bit as a sign: 0 is positive, 1 is negative
   - 0111 1111 corresponds to 127
   - 1111 1111 corresponds to -127
   - Problems:
     - zero is represented twice, 1000 0000 and 0000 0000
     - Not possible to directly use this representation in sums

2. Two’s complement
   - represent positive numbers up to 127 normally
   - represent negative numbers as the positive, negated (bit by bit) plus 1
   
   Example: represent $-58$ on 8 bits:
   - 58 is: 0011 1010
   - negation is: 1100 0101
   - plus 1: 1100 0110

   Hence, the representation of $-58$ is 1100 0110
Advantages of two’s complement

- Range with 9 bits: (−128 ; +127)
- By summing positive and negative numbers using two’s complement representation, the result is correct if it is in range

Example 1:
- −58 is 1100 0110
- 64 is: 0100 0000

\[
\begin{align*}
\text{Sum} & \quad 1100\ 0110 \quad + \\
& \quad 0100\ 0000 \quad = \\
& \quad 0000\ 0110 \quad (6)
\end{align*}
\]

Example 2:
- −58 is 1100 0110
- 32 is: 0010 0000

\[
\begin{align*}
\text{Sum} & \quad 1100\ 0110 \quad + \\
& \quad 0010\ 0000 \quad = \\
& \quad 1110\ 0110 \quad (-26)
\end{align*}
\]
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The challenge is to represent everything with just two symbols

- Numbers, text, drawings, images, pictures, sounds, complex music, etc.

Let’s start with numbers:

- We already know that with 8 bits we can represent integers from $-128$ to $127$
- with 16 bits (2 bytes) we can represent from $-2^{15}(-32768_{b(10)})$ to $2^{15} - 1(32767_{b(10)})$
- with 32 bits (4 bytes) we represent from $-2^{31}(-2,147,483,648_{b(10)})$ to $2^{31} - 1(2,147,483,647_{b(10)})$
- with 64 bits (8 bytes) we represent up to $2^{63} - 1$ which is $9,223,372,036,854,775,807_{b(10)} \approx 10^{20}$ (approximately the number of grains of sand on earth)
Two possible systems:

- **Fixed point representation**: a fixed number of bits are for the integer part, the remaining for the rational part
  - Used in some embedded system (DSP) because calculations are usually faster
  - fixed precision, limited range

- **Floating point representation**: a fixed number of bits to represent the mantissa, and the remaining to represent the exponent
  - Used in modern PCs
  - very wide range, variable precision
IEEE 754 standard for floating point

Figure: Floating point, single precision

Figure: Floating point, double precision
Symbols and meaning

- Obviously, symbols are meaningless *per-se*.
- The meaning we attach to them depends on the context.
- A string of 64 bits can be:
  - a large integer, two smaller 32-bit integers, four 16-bits integers, eight 8-bits integers
  - or, two 32-bits double precision floating point numbers
  - or, four 16 bits single precision floating point numbers
  - or ...?
It is possible to represent characters and strings of characters using an appropriate encoding.

The ASCII encoding assigns each character a number between 0 and 255.

Some example of character encoding:

<table>
<thead>
<tr>
<th>bin</th>
<th>dec</th>
<th>glyph</th>
<th>bin</th>
<th>dec</th>
<th>glyph</th>
</tr>
</thead>
<tbody>
<tr>
<td>011 0000</td>
<td>48</td>
<td>’0’</td>
<td>110 0001</td>
<td>97</td>
<td>a</td>
</tr>
<tr>
<td>011 0001</td>
<td>49</td>
<td>’1’</td>
<td>110 0010</td>
<td>98</td>
<td>b</td>
</tr>
<tr>
<td>011 0010</td>
<td>50</td>
<td>’2’</td>
<td>110 0011</td>
<td>99</td>
<td>c</td>
</tr>
<tr>
<td>011 0011</td>
<td>51</td>
<td>’3’</td>
<td>110 0100</td>
<td>100</td>
<td>d</td>
</tr>
<tr>
<td>011 0100</td>
<td>52</td>
<td>’4’</td>
<td>110 0101</td>
<td>101</td>
<td>e</td>
</tr>
<tr>
<td>011 0101</td>
<td>53</td>
<td>’5’</td>
<td>110 0110</td>
<td>102</td>
<td>f</td>
</tr>
<tr>
<td>011 0110</td>
<td>54</td>
<td>’6’</td>
<td>110 0111</td>
<td>103</td>
<td>g</td>
</tr>
<tr>
<td>011 0111</td>
<td>55</td>
<td>’7’</td>
<td>110 1000</td>
<td>104</td>
<td>h</td>
</tr>
<tr>
<td>011 1000</td>
<td>56</td>
<td>’8’</td>
<td>110 1001</td>
<td>105</td>
<td>i</td>
</tr>
<tr>
<td>011 1001</td>
<td>57</td>
<td>’9’</td>
<td>110 1010</td>
<td>106</td>
<td>j</td>
</tr>
</tbody>
</table>
Representing text

- A simple text:

  *This course is valid 3 credits.*

- And its representation

<table>
<thead>
<tr>
<th>T</th>
<th>h</th>
<th>i</th>
<th>s</th>
<th>c</th>
<th>o</th>
<th>u</th>
</tr>
</thead>
<tbody>
<tr>
<td>01010100</td>
<td>01101000</td>
<td>01101001</td>
<td>01110011</td>
<td>00010000</td>
<td>01100011</td>
<td>01101111</td>
</tr>
<tr>
<td>r</td>
<td>s</td>
<td>e</td>
<td>i</td>
<td>s</td>
<td>v</td>
<td></td>
</tr>
<tr>
<td>01110010</td>
<td>01110011</td>
<td>01100101</td>
<td>00010000</td>
<td>01101001</td>
<td>01110011</td>
<td>00010000</td>
</tr>
<tr>
<td>a</td>
<td>l</td>
<td>i</td>
<td>d</td>
<td>3</td>
<td>c</td>
<td></td>
</tr>
<tr>
<td>01100001</td>
<td>01101100</td>
<td>01101001</td>
<td>01100100</td>
<td>00010000</td>
<td>00110011</td>
<td>00010000</td>
</tr>
<tr>
<td>r</td>
<td>e</td>
<td>d</td>
<td>i</td>
<td>t</td>
<td>s</td>
<td></td>
</tr>
<tr>
<td>01110010</td>
<td>01100101</td>
<td>01100100</td>
<td>01101001</td>
<td>01110100</td>
<td>01110011</td>
<td>00101110</td>
</tr>
</tbody>
</table>

**Figure:** A string of text, and its binary representation.
Other characters

- There are many characters in the world
  - Chinese, Japanese, Hindu, Arabic, ...  
- The new standard Unicode covers all possible characters, needs 16 bits per character
- An Unicode representation of the same text will take double the space of the corresponding ASCII encoding (16 bits per character, instead of 8 bit per character)
- It is possible to compact a text by using an appropriate compression algorithm (tries to avoid repetition of symbols by using a different (and more compact) encoding)
Representing signals

- With bits and bytes we can represent numbers.
- To represent functions (i.e., signal), we can store a sequence of numerical values.
  - For example, to represent music, we can store one function $f_i(t)$ for each instrument.
  - Since the function is continuous, we first sample it in small intervals of time.

**Figure: Sampling**
Representing music

- Like in the case of the text, raw music representation has a lot of redundant information, and take a lot of space.
- For example, 1 msec sampling, each value with 32 bits, means approximately 4 Kb per second per channel.

<table>
<thead>
<tr>
<th>File Type</th>
<th>44.1 Khz</th>
<th>22.05 Khz</th>
<th>11.025 Khz</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 bit stereo</td>
<td>10.1</td>
<td>5.05</td>
<td>2.52</td>
</tr>
<tr>
<td>16 bit mono</td>
<td>5.05</td>
<td>2.52</td>
<td>1.26</td>
</tr>
</tbody>
</table>

Table: Memory requirement for 1 minute of Digital Audio (all numbers in Mbytes)

- Of course, there is a tradeoff between sampling rate and quality.
  - See [http://www.cs.cf.ac.uk/Dave/Multimedia/node150.html](http://www.cs.cf.ac.uk/Dave/Multimedia/node150.html) for a comparison

- It is possible to compress such representation by using appropriate encoding algorithms (e.g. mp3, ogg, etc.), although some quality gets lost.
Something similar is done with pictures

- A picture is first divided into *pixels*
- Each pixel is represented as a number or a set of numbers
- Most common representation is RGB (Red-Green-Blue)
- By using 8 bits for each of the three colors, each pixel is represented by 24 bits
  - A 1024x800 image is large $3 \times 1024 \times 800 \approx 2$ Mbytes.
- Of course, it is possible to compress pictures as well

Finally, movies are just sequences of pictures. Here compression is utterly necessary!
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Representing information

- At the end, every information is coded as a sequence of just two symbols: 0 and 1
- A processor just acts on such two symbols to perform any kind of computation
- How does a processor know what to do?
- Processors are *programmable* machines
- They take
  1. A *program*, i.e. a sequence of *instructions* (the recipe!)
  2. any sequence of bits as input,
  3. and perform *transformations* (computations) on this sequence according to the program, to produce a sequence of bits in output
- In the next, we will give an overview of how this process works
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Questions

1. How would you represent rational numbers?
   - A rational number is for example $\frac{1}{3}$. It cannot be finitely represented as a decimal number because it has infinite digits $1.3333\ldots$.

2. Is there any way to represent irrational numbers with infinite precision?
   - For example $\sqrt{2}$, $e$, $\pi$, etc.

3. Is there a way to represent an integer number with an arbitrary large number of digits?