Fundamentals of Programming

Finite State Machines

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Introduction

State machines are basic building blocks for computing theory.

- very important in theoretical computer science
- many applications in practical systems
- There are many slightly different definitions, depending on the application area
- A state machine is a Discrete Event Discrete State system
  - transitions from one state to another only happen on specific events
  - events do not need to occur at specific times
  - we only need a temporal order between events (events occur one after the other), not the exact time at which they occur

Definition

A deterministic finite state machine (DFSM) is a 5-tuple:

- S (finite) set of states
- I set of possible input symbols (also called input alphabet)
- s_0 initial state
- ϕ transitions: a function from (state,input) to a new state

$$\phi : S \times I \rightarrow S$$

- ω output function (see later)

An event is a new input symbol presented to the machine.

- In response, the machine will react by updating its state and possibly producing an output. This reaction is istantaneous (synchronous assumption).

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Output function

Two types of machines:

- **Moore** output only depends on state:

$$\omega_{mr} : S \rightarrow \Omega$$

  Where Ω is the set of output symbols. In this case, the output only depends on the state, and it is produced upon entrance on a new state.

- **Mealy** output depends on state and input:

$$\omega_{ml} : S \times I \rightarrow \Omega$$

  In this case, the output is produced upon occurrence of a certain transaction.

Moore machines

- Moore machines are the simplest ones
- If $\Omega = \{\text{yes}, \text{no}\}$, the machine is a recognizer
- A recognizer is able to accept or reject sequences of input symbols
- The set of sequences accepted by a recognizer is a regular language

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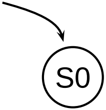
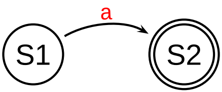
State diagrams

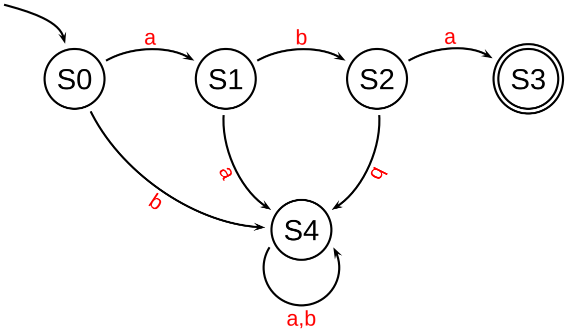
- FSM can be represented by State Diagrams





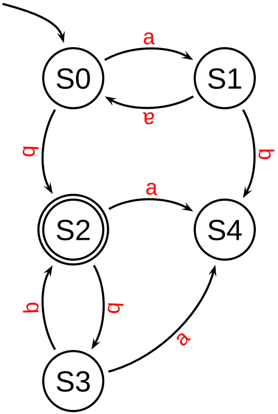
- final states are identified by a double circle

- In this example $I = \{a, b\}$. The following state machine recognizes string *aba*



Example: recognizer II

- Recognize string $a^n b^m$ with n even and m odd (i.e. *aabbb*, *b*, *aab* are all legal sequences, while *a*, *aabb*, are non legal)



- S4 is an error state. It is not possible to go out from an error state (for every input, no transaction out of the state)
- S2 is an accepting state, however we do not know the length of the input string, so it is possible to exit from the accepting state if the input continues
- If we want to present a new string we have to reset the machine to its initial state

Non regular language

- FSM are not so powerful. They can only recognize simple languages
- Example:
  - strings of the form $a^n b^n$ for all $n \geq 0$ cannot be recognized by a FSM (because they only have a finite number of states)
  - they could if we put a limit on n. For example, $0 \leq n \leq 10$.

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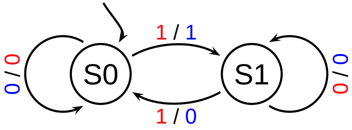
Mealy machines

- In Mealy machines, output is related to both state and input.
- In practice, output can be associated to a transition
- Given the synchronous assumption, the Moore's model is equivalent to the Mealy's model: for every Moore machine, it is possible to derive an equivalent Mealy machine, and viceversa

Example: parity check

- In this example, we have a Mealy machine that
  - outputs 1 if the number of symbols 1 in input so far is odd;
  - it outputs 0 otherwise.



- Usually, Mealy machines have a more compact representation than Moore machines (i.e. they perform the same task with a number of states that is no less than the equivalent Moore machine).

Table representation

- A FSM can be represented through a table
- The table shown below corresponds to the parity-check Mealy FSM shown just before.

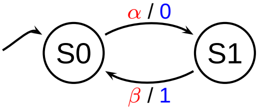
	0	1
S_0	S_0 / 0	S_1 / 1
S_1	S_1 / 1	S_0 / 0

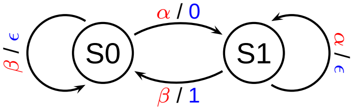
Stuttering symbol

- Input and output alphabets include the absent symbol ϵ
- It correspond to a null input or output
- When the input is absent, the state remains the same, and the output is absent
- Any sequence of inputs can be interleaved or extended with an arbitrary number of absent symbols without changing the behavior of the machine
- the absent symbol is also called the stuttering symbol

Abbreviations

- If no guard is specified for a transition, the transition is taken for every possible input (except the absent symbol ϵ)
- If no output is specified for a transition, the output is ϵ
- given a state S_0, if a symbol α is not used as guard of any transition going out of S_0, then an implicit transition from S_0 to itself is defined with α as guard and ϵ as output



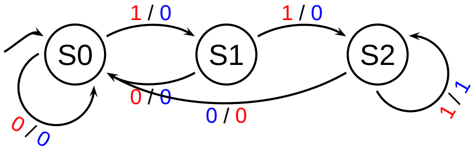


Exercise

- Draw the state diagram of a FSM with $I = \{0, 1\}$, $\Omega = \{0, 1\}$, with the following specification:
  - let $x(k)$ be the sequence of inputs
  - the output $\omega(k) = 1$ iff $x(k-2) = x(k-1) = x(k) = 1$

Solution

- three states: $S0$ is the initial state, $S1$ if last input was 1, $S2$ if last two inputs were 1



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Deterministic machines

- Transitions are associated with
  - a source state
  - a guard (i.e. a input value)
  - a destination state
  - a output
- in deterministic FSM, a transition is uniquely identified by the first two.
- in other words, given a source state and a input, the destination and the output are uniquely defined

Non deterministic FSMs

- A non deterministic finite state machine is identified by a 5-tuple:
  - I set of input symbols
  - Ω set of output symbols
  - S set of states
  - S_0 set of initial states
  - ϕ transition function:

$$\phi : S \times I \rightarrow (S \times \Omega)^*$$

    where S^* denotes the power set of S, i.e. the set of all possible subsets of S.

- In other words, given a state and an input, the transition returns a set of possible pairs (new state, output).

Non determinism

- Non determinism is used in many cases:
  - to model randomness
  - to build more compact automata

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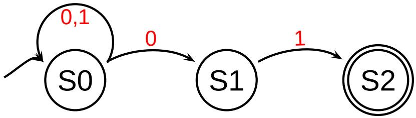
- Randomness has nothing to do with probability! we do not know the probability of occurrence of every behavior, we only know that they are possible
- A more abstract model of a system hides *unnecessary* details, and it is more compact (less states)

Example of non deterministic state machine

- We now build an automata to recognize all input strings (of any lenght) that end with a `01`



Equivalence between D-FSM and N-FSM

- It is possible to show that Deterministic FSMs (D-FSMs) are equivalent to non deterministic ones(N-FSMs)
- Proof sketch
  - Given a N-FSM $\mathcal{A}$, we build an equivalent D-FSM $\mathcal{B}$ (i.e. that recognizes the same strings recognized by the N-FSM. For every subset of states of the $\mathcal{A}$, we make a state of $\mathcal{B}$. Therefore, the maximum number of states of $\mathcal{B}$ is $2^{|S|}$. The start state of $\mathcal{B}$ is the one corresponding to the $\mathcal{A}$. For every subset of states that are reachable from the start state of state of $\mathcal{A}$ with a certain symbol, we make one transition in $\mathcal{B}$ to the state corresponding to the sub-set. The procedure is iterated until all transitions have been covered.

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Exercise

- As an exercise, build the D-FSM equivalent to the previous example of N-FSM

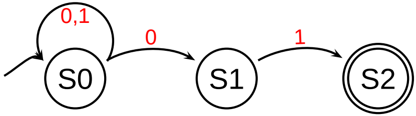


Figure: The N-FSM

Solution

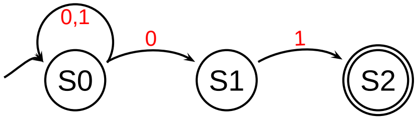


Figure: The N-FSM

- Initial state: $\{S0\}$

state name	subset	0	1
q0	{S0}	{S0, S1}	{S0}
q1	{S0,S1}	{S0, S1}	{S0, S2}
q2	{S0,S2}	{S0, S1}	{S0}

Solution

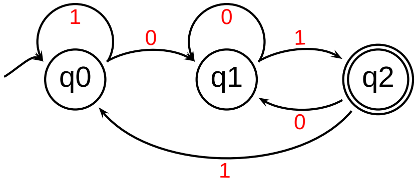


Figure: The equivalent D-FSM

Outline

Encoding states and symbols

- The first thing to do is to encode states and event symbols in the state machine.
- States can be simply and enumerated, so they can be represented by an integer variable

```
#define STATE_1   1
#define STATE_2   2
#define STATE_3   3
...
int current_state;
...
```

- Same thing can be done for the events

```
#define EVENT_1    1
#define EVENT_2    2
#define EVENT_3    3
...
int event;
...
```

Enum in C

- In C, an enumerated type can also be defined with the keyword `enum`

```
enum states { STATE_1, STATE_2, STATE_3, MAX_STATES } current_state;
enum events { EVENT_1, EVENT_2, MAX_EVENTS } new_event;
```

- The C compiler maps those variables into `int`
- Therefore, it is just a notation, no new added feature

Actions

- The main cycle is the following:
- When an even arrives
  - check the current state
  - depending on the event, perform an action and change state
- A simple way to perform this is through a sequence of `if-then-else` or `switch-case`

```
switch(current_state) {
case STATE_1 : if (new_event == EVENT_1) {
                  // action for EVENT_1
                  // change state
               } else if (new_event == EVENT_2) {
                  // new action for EVENT_2
                  // change state
               } ...
case STATE_2  : if (new_event == EVENT_1) {
                  // action for EVENT_1
                  // change state
                }
...
}
```

Functions

- The previous implementation does not scale for large number of states and events
- A more modular implementation consists in having one separate function per action

```
void action_s1_e1 ();
void action_s1_e2 ();
void action_s2_e1 ();
void action_s2_e2 ();
void action_s3_e1 ();
void action_s3_e2 ();
```

- In this way, functions can go in separate files

```
void action_s1_e1 ()
{
    /* do some processing here */
    current_state = STATE_2; /* set new state, if necessary */
}
```

Function Table

- All functions can be stores in a table of states-events

```
typedef void (*ACTION)();

ACTION [MAX_STATES][MAX_EVENTS] = {
    { action_s1_e1, action_s1_e2 }, /* procedures for state 1 */
    { action_s2_e1, action_s2_e2 }, /* procedures for state 2 */
    { action_s3_e1, action_s3_e2 }  /* procedures for state 3 */
};
```

- Of course, not all transitions are possible
- In this case, you can define empty functions, or functions that return an error

Main cycle

- The main program cycle is then:

```
new_event = get_new_event (); /* get the next event to process */

if (((new_event >= 0) && (new_event < MAX_EVENTS))
    && ((current_state >= 0) && (current_state < MAX_STATES))) {
    /* call the action procedure */
    state_table[current_state][new_event] ();

} else {
    /* invalid event/state - handle appropriately */
}
```

Consideration

- In the previous implementation, all functions act on a global variable `current_state` to modify the state
- To make the implementation less dependent on a global variable, it is possible to pass the state and event, and return the new state variable;

```
typedef int (*ACTION)(int, int);

int action_s1_e1(int state, int event) {
    /* do something */
    return STATE_2; /* returns the new state */
}
```

- In this way, it is also possible to write more functions that can be reused for different states and events

Outline

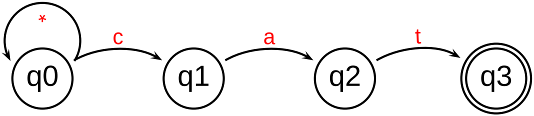
Regular expressions and automata

- Regular expressions are equivalent to Finite State Automata
  - In fact, a regular expression can be translated to an automaton, by means of an appropriate parser
  - This is exactly what the `grep` program does
- Regular expression syntax (POSIX)

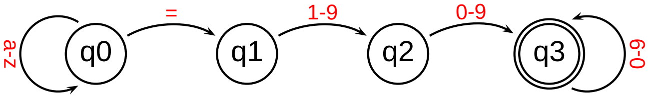
Symbol	meaning
`.`	matches any single character
`[abc]`	matches any of the characters within the brackets. [a-z] specifies a range which matches any lowercase letter from “a” to “z”
`[^ abc]`	matches any of the characters not within the brackets
`^`	matches the starting position of the input line
`$`	matches the ending position of the input line
`*`	matches the preceding character or expression any number of times (including 0)
+	matches the preceding character or expression one or more number of times
?	matches the preceding character or expression zero or one times
|	Or between two expressions

Examples

- `*cat*` any string containing the substring cat



- `[a-z]*=[1-9][0-9]*` An assignment (for example $x = 50$)



- (Here I am assuming that characters not in the event list will abort the machine)

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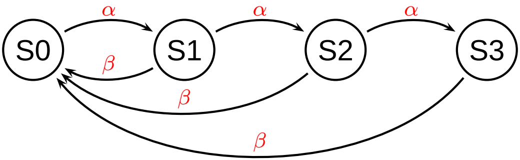
Problems with FSMs

- FSM are *flat* and *global*
- All states stay on the same level, and a transition can go from one state to another
  - It is not possible to *group* states and transitions
- Replicated transition problem:



Product of two FSM

- Another problem is related to the cartesian product of two FSM
  - Suppose we have two distinct FSMs that we want to combine into a single one

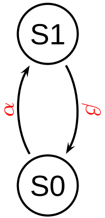


Figure: FSM 1

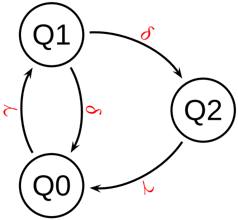
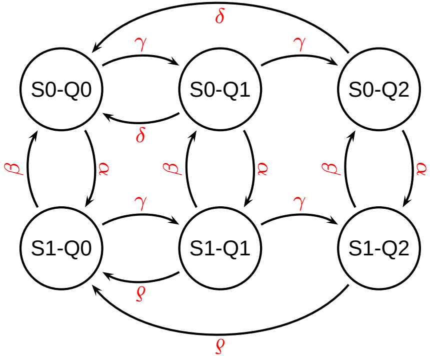


Figure: FSM 2

Product result

- The result is a state machine where each state corresponds to a pair of states of the original machines
- Also, each transition in corresponds to one transition in either of the two original state machines



Complexity handling

- All these problems have to do with *complexity* of dealing with states
- In particular, the latter problem is very important, because we often need to combine different simple state machines
- However, the resulting diagram (or table specification) can become very large
- We need a different specification mechanism to deal with such complexity
- In this course, we will study Statecharts (similar to Matlab StateFlow), first proposed by Harel

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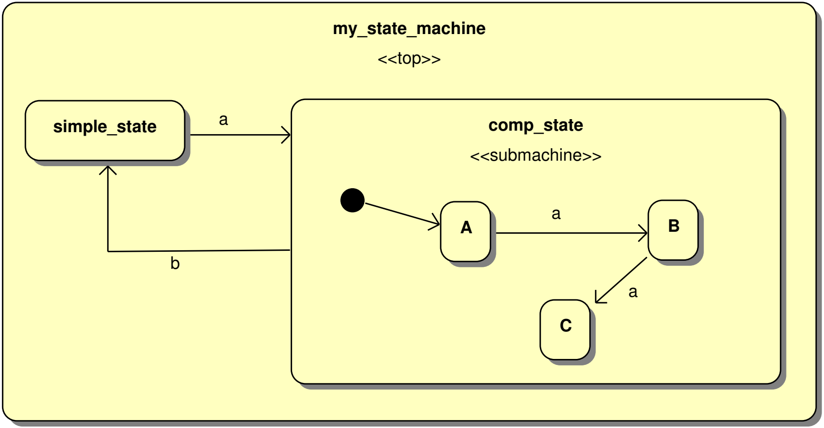
States

- In H-FSMs, a state can be final or composite



State specification

- A state consist of:
  - An *entry* action, executed once when the system enters the state
  - An *exit* action, executed once before leaving the state
  - A *do* action, executed *while* in the state (the semantic is not very clear)
- They are all optional

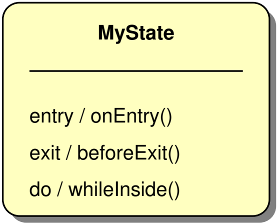


Figure: Entry, exit and do behaviors

Transitions

- A transition can have:
  - A *triggering event*, which activates the transition
  - A *guard*, a boolean expression that *enables* the transition. If not specified, the transition is always enabled
  - An *action* to be performed if the transition is activated and enabled, just after the exit operation of the leaving state, and before the entry operation of the entering state
- Only the triggering event specification is mandatory, the other two are optional

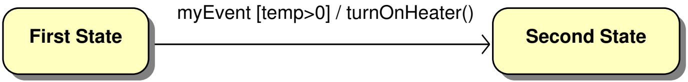
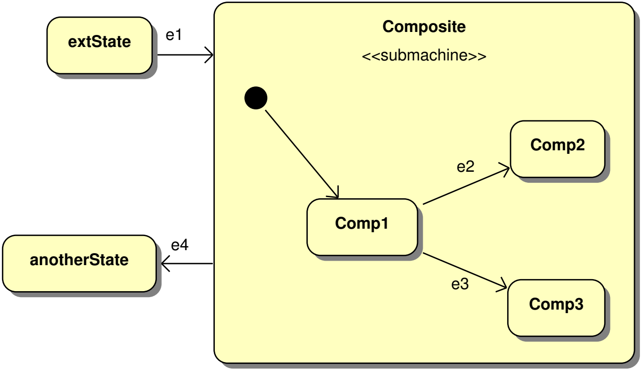


Figure: Transition, with event, guard and action specified

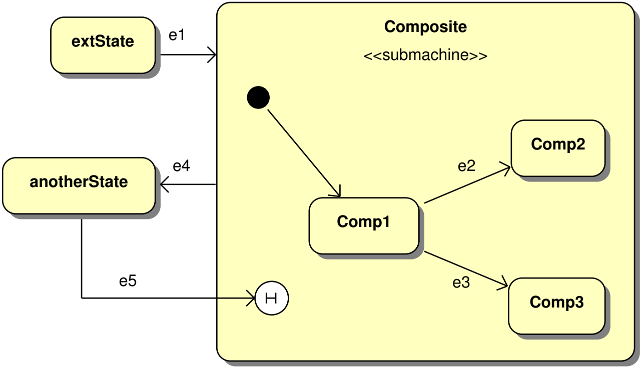
Or composition

- A state can be decomposed into substates
- When the machine enters state *Composite*, it goes into state *Comp1*
- Then, if event *e2* it goes in *Comp2*, if event *e3* it goes in *Comp3*, else if event *e4* it exits from *Composite*.



History

- When the machine exits from a composite state, normally it *forgets* in which states it was, and when it enters again, it starts from the starting state
- To “remember” the state, so that when entering again it will go in the same state it had before exiting, we must use the *history* symbol



AND decomposition

- A state can be decomposed in orthogonal regions, each one contains a different sub-machine
- When entering the state, the machine goes into one substate for each sub-machine

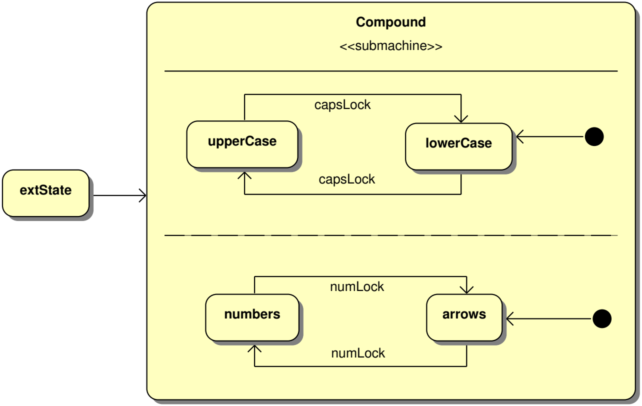


Figure: Orthogonal states for a keyboard

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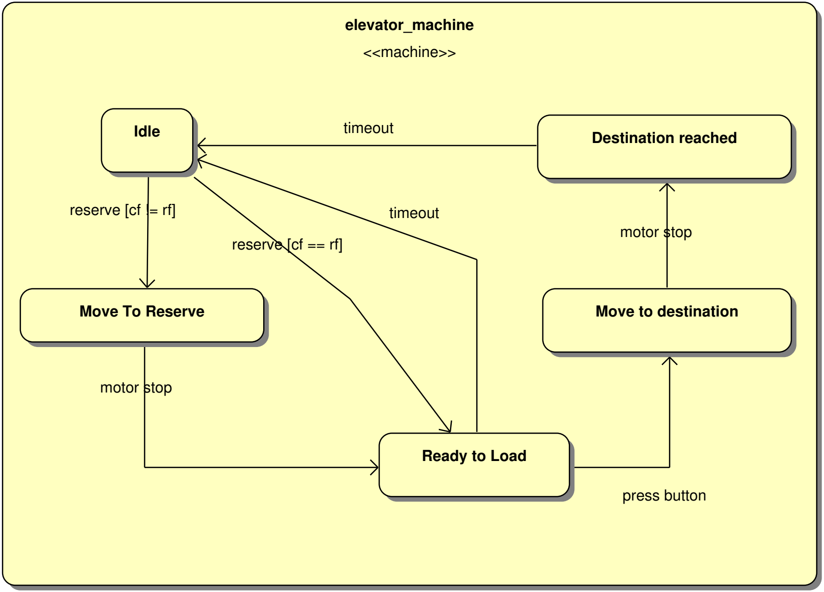
Elevator

- Let's define an "intelligent" elevator
  - For a 5-stores building (ground floor, and four additional floors)
  - Users can "reserve" the elevator
  - The elevator serves all people in order of reservation
- We assume at most one user (or group of users) per each "trip", and they all need to go to the same floor

Design considerations

- How do you encode at which floor the elevator is?
  1. One different state per each floor
     - Does not scale well; for 100 floors bulding, we need 100 states!
  2. The floor is encoded as an *extended* state, i.e. a variable `cf`
     - It scales, but more difficult to design
  3. It always depends on what we want to describe!
- Which events do we have?
  - An user press a button to “reserve” the elevator, setting variable `rf`
  - An user inside the elevator presses the button to change floor, setting variable `df`

First design



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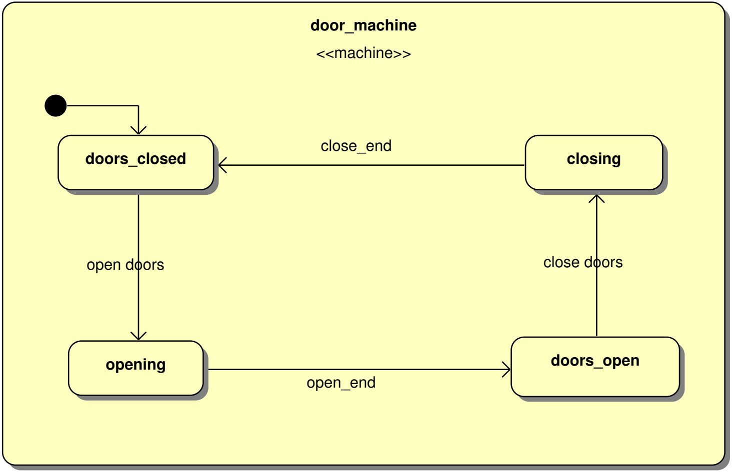
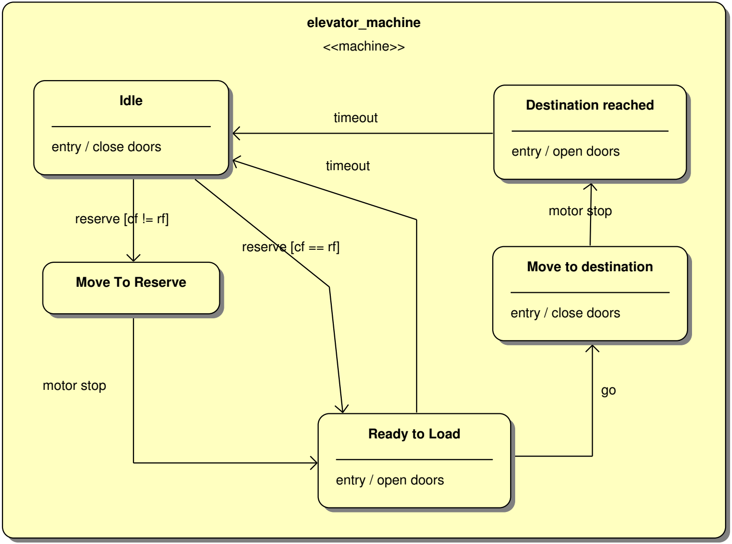
Doors

- The previous design does not capture all aspects of our systems
- Let's start to add details by adding the description of how the doors behave
- Abstraction level
  - The level of details of a design depends on what the designer is more interested in describing with the specification
  - In the previous design, we were not interested in describing all aspects, but only on giving a few high-level details
  - The design can be refined by adding details when needed

The doors submachine



The elevator, second design



Putting everything together