

Ingegneria dell'Automazione - Sistemi in Tempo Reale

*Selected topics on discrete-time and
sampled-data systems*

Luigi Palopoli

palopoli@sssup.it - Tel. 050/883444

Outline

- Systems and I/O behaviours
- Linear time-invariant Discrete-time systems and behaviours
 - Discrete Transfer-function
 - State-space representations
- Representation of a sampled-data system

Definitions

- A timeset $\mathcal{T}$ is a subgroup of $(\mathbb{R}, +)$
 - Continuous-Time: $\mathcal{T} = \mathbb{R}$
 - Discrete-Time (DT): $\mathcal{T} = \mathbb{Z}$
 - Discrete-Event: $\mathcal{T}$ is a countable subset of $\mathbb{R}$ and there is a finite number of elements between any two elements

Sequences

- a sequence u is a mapping from a subset of $\mathcal{T}$ to a set $\mathcal{U}$
- given an interval I the set of all sequences from I into $\mathcal{U}$ is denoted as:

$$\mathcal{U}^I = \{\omega \mid \omega : I \rightarrow \mathcal{U}\}$$

- Example if $\mathcal{T} = \mathbb{Z}$ the set $\mathcal{U}^{[0, k)} = \omega(0), \dots, \omega(k - 1)$

System

- a system (or machine) is a five tuple $\Sigma = (\mathcal{T}, \mathcal{X}, \mathcal{U}, \phi)$ consisting of:
 - a timeset $\mathcal{T}$
 - a nonempty statespace $\mathcal{X}$
 - an input-value space $\mathcal{U}$
 - a transition map $\phi : D_\phi \rightarrow \mathcal{X}$ defined on a subset D_ϕ of:

$$\{(\tau, \sigma, x, \omega) \mid \tau, \sigma \in \mathcal{T}, x \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma, \tau)}\}$$

- ... and some technical axioms that we omit here.

System with output

- A system with output is a system Σ with:
 - a set $\mathcal{Y}$ called the measurement-value space
 - a readout map h :

$$h : \mathcal{T} \times \mathcal{X} \rightarrow \mathcal{Y}$$

- it is possible to consider output maps where the output depends also on the input (e.g., non strictly causal systems)

$$h : \mathcal{T} \times \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}$$

Some taxonomy

- a DT system is one for which $\mathcal{T} = \mathbb{Z}$
- a system is time-invariant (TI) iff

*for each $\omega \in \mathcal{U}^{[\sigma, \tau)}$, each $x \in \mathcal{X}$, each $\mu \in \mathcal{T}$,
if ω is admissible for x then $\omega^\mu \in \mathcal{U}^{[\sigma + \mu, \tau + \mu)}$, $\omega^\mu =$
 $\omega(t - \mu)$ is admissible for x ,
moreover $\phi(\tau, \sigma, x, \omega) = \omega(t + \mu, \sigma + \mu, x, \omega^\mu)$*

FSM

- Finite State Machines (FSM)
 - $\mathcal{X}$ is finite
 - $\mathcal{U}$ is finite
 - $\mathcal{Y}$ is finite
 - $\mathcal{T}$ is a discrete-event timeset

Trajectory

- A trajectory is a pair of functions (ξ, ω) , $\xi \in \mathcal{X}^I$, $\omega \in \mathcal{U}^I$ such that

$$\xi(\tau) = \phi(\tau, \sigma, \xi(\sigma), \omega|_{[\sigma, \tau)})$$

holds for each pair $\sigma, \tau \in I, \sigma < \tau$

I/O behaviour

- an I/O behaviour is a 4-tuple $\Lambda = (\mathcal{T}, \mathcal{U}, \mathcal{Y}, \lambda)$ consisting of
 - a timeset $\mathcal{T}$
 - a nonempty control-value set $\mathcal{U}$
 - a nonempty output-value set $\mathcal{Y}$
 - a response map: $\lambda : D_\lambda \rightarrow \mathcal{Y}$ which is defined on a nonempty subset D_λ of

$$\{(\tau, \sigma, \omega) \mid \sigma, \tau \in \mathcal{T}, \sigma \leq \tau, \omega \in \mathcal{U}^{[\sigma, \tau)}\}$$

I/S behaviour - System

- Initialised system is a pair (Σ, x_0) , where x_0 is the initial state
- an Input/State (I/S) behaviour of (Σ, x_0) is the behaviour with the same $\mathcal{T}, \mathcal{U}$ as Σ and with output-value space $\mathcal{X}$, whose response map is defined on the projection

$$\{(\tau, \sigma, \omega) \mid (\tau, \sigma, x_0, \omega) \in D_\phi\}$$

$$\lambda(\tau, \sigma, \omega) = \phi(\tau, \sigma, x_0, \omega).$$

I/O behaviour of a System

- if (Σ, x_0) is an initialised system with output, then the I/O behaviour of (Σ, x_0) has the same $\mathcal{T}, \mathcal{U}, \mathcal{Y}$ as σ
- the domain is the projection

$$\{(\tau, \sigma, \omega) \mid (\tau, \sigma, x_0, \omega) \in D_\phi\}$$

- and response map:

$$\lambda(\tau, \sigma, \omega) = h(\tau, \phi(\tau, \sigma, x_0, \omega))$$

Linear systems

- A DT system Σ is linear over the field $\mathbb{K}$ if
 - it is complete (every input is admissible for every state)
 - $\mathcal{X}$ and $\mathcal{U}$ are vector spaces
 - $\mathcal{P}(t, ., .)$ is linear for each $t \in \mathbb{Z}$ where

$$\mathcal{P}(t, x, u) = \phi(t + 1, t, x, \omega)$$

- in addition, if the system has output then
 - $\mathcal{Y}$ is a vector space
 - $h(t, .)$ is linear for each t

Linear DT systems

- Linearity is equivalent to the existence of two matrices $A(t)$, $B(t)$ such that

$$\mathcal{P}(t, x, u) = A(t)x + B(t)u$$

- if the system has outputs then

$$h(t, x) = C(t)x$$

- if the output depends on u then

$$h(t, x, u) = C(t)x + D(t)u$$

- if the system is TI then $A(t)$, $B(t)$, $C(t)$, $D(t)$ do not depend on t

Linear DT I/O behaviours

- An I/O behaviour Λ is linear if:
 - it is complete
 - $\mathcal{U}, \mathcal{Y}$ are vector spaces
 - for each $\sigma \in \mathbb{Z}$ and $\tau \in \mathbb{Z}$, with $\sigma \leq \tau$, $\lambda(\tau, \sigma, \omega)$ is linear with respect to ω

Linear DT TI I/O behaviours - I

- introduce $\delta(t)$ (kronecker delta): $\delta(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$

- a generic sequence $u(t)$ can be expressed as

$$u(t) = \sum_{k=\sigma}^{\tau} u(k) \delta(t - k)$$

- let $h(t) = \lambda(t, 0, \delta(t))$ (pulse response)

- due to linearity and time invariance, the response to

$u(t)$ can be expressed as

$$y(t) = \sum_{k=\sigma}^{\tau} h(t - k) u(k) \triangleq h(k) * u(k) \text{ (convolution).}$$

For general sequences and noncausal systems

$$\sigma = -\infty, \tau = +\infty$$

The Z –transform

- The Z –transform plays for DT systems as the Laplace transform plays in CT systems
- given a sequence $e(k)$, the Z – transform $E(z)$ is given by $E(z) = \mathcal{Z}[e(t)] = \sum_{-\infty}^{+\infty} e(k)z^{-k}$ where z is a complex variable

The Z –transform I

● Some properties

- Linearity: $\mathcal{Z}[\alpha f + \beta g] = \alpha F(z) + \beta G(z)$

- Time-shift: $\mathcal{Z}[f(t - n)] = z^{-n} F(z)$

- Convolution:

$$y(t) = h(t) * u(t) \Rightarrow Y(z) = H(z)U(z)$$

- Forward step: $\mathcal{Z}[y(t + 1)] = z(Y(z) - y_0)$

- ...please consult a basic text book...

Transfer function

- General notations for expressing the transfer function (TF), i.e., the Z –transform of the pulse response

- Polynomial representation: $H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}$

Matlab code:

```
num = [b0, b1, ...bm]; den  
= [1, a1, ...an];  
sys = tf(num,den,T);
```

- Zero Pole representation: $H(z) = K \frac{\prod_{i=1}^m (z - z_i)}{\prod_{i=1}^n (z - p_i)}$

Matlab code:

```
z = [z1, ...bm]; p= [p1  
...pn]; k = K;  
sys = zpk(z,p,k,T);
```

Zeros and poles

- Poles are the zeros of the denominator. A pole p_i corresponds to a free mode of the system associated with the time function $z = p_i^t$
- Zeros correspond to blocking properties of the system: $z = z_i$ means that there exist initial conditions such that the system has response 0 to the signal z_i^t

Zeros and poles - I

● Example:

$$y(t+1) - ay(t) = u(t+1) - bu(t)$$

● Computing the $\mathcal{Z}$ -transform we get:

$$Y(z) = \frac{z}{z-a}(y_0 - u_0) + \frac{z-b}{z-a}U(z)$$

● $u(t) = b^t \Rightarrow U(z) = \frac{z}{z-b}$

● Therefore

$$Y(z) = \frac{z}{z-a}(y_0 - 1)$$

● if $y_0 = 1$ then $Y(z) = 0$

Zeros and poles - II

- A more subtle example:

$$y(t + 1) - ay(t) = u(t + 1) - au(t)$$

The transfer function is $\frac{z-a}{z-a}$; is it equivalent to 1?

Zeros and poles - II

- A more subtle example:

$$y(t + 1) - ay(t) = u(t + 1) - au(t)$$

The transfer function is $\frac{z-a}{z-a}$; is it equivalent to 1?

- NO! The evolution of the system is given by:

$$y(t) = a^t y_0 + u(t)$$

Zeros and poles - II

- A more subtle example:

$$y(t + 1) - ay(t) = u(t + 1) - au(t)$$

The transfer function is $\frac{z-a}{z-a}$; is it equivalent to 1?

- NO! The evolution of the system is given by:

$$y(t) = a^t y_0 + u(t)$$

- Problems are limited only if $|a| < 1$

TF of a LTI DT system

- consider an initialised Single Input Single Output (SISO) system

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \\ x_0 &= 0\end{aligned}\quad \text{with initial state}$$

- the pulse response is $y(t) = \sum_{j=0}^t CA^jB \rightarrow$
 $Y(z) = C(zI - A)^{-1}BU(z)$

Realization of a TF

- Realization is the converse problem: given a TF $H(z)$ find a system Σ whose TF is $H(z)$
- Preliminary point: is a realization unique?
NO!, Given a realization A, B, C also A', B', C' is a realization, where

$$A' = \begin{bmatrix} a & 0 \\ 0 & A \end{bmatrix}, B' = \begin{bmatrix} 0 \\ b \end{bmatrix}, C' = [h \ C],$$

for any a, h .

- we aim at *minimal* realization (completely controllable and observable)

Control canonical form

- Start from: $U(z) = H(z)E(z) = \frac{a(z)}{b(z)}E(z),$

where $a(z) = z^3 + a_1z^2 + a_2z + a_3,$

$$b(z) = b_0z^3 + b_1z^2 + b_2z + b_3$$

- Introduce

$$\xi = E(z)/a(z) \rightarrow U(z) = b(z)\xi, \quad a(z)\xi = E(z)$$

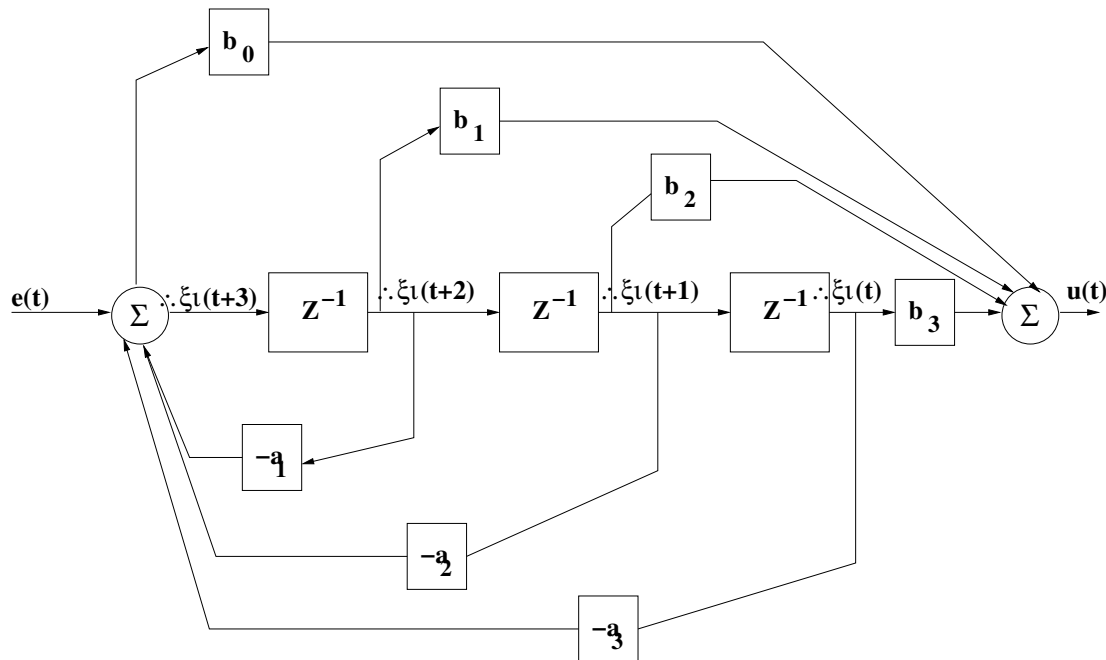
- From the properties of the Z -transform:

$$a(z)\xi = E(z) \Rightarrow \xi(t+3) = e(t) - a_1\xi(t+2) - a_2\xi(t+1) - a_3\xi(t)$$

$$b(z)\xi = U(z) \Rightarrow u(t) = b_0\xi(t+3) + b_1\xi(t+2) + b_2\xi(t+1) + b_3\xi(t)$$

Control canonical form - I

🔴 Graphical representation:



Control canonical form - II

- Define $\xi(t+2) = x_1(t)$, $\xi(t+1) = x_2(t)$, $\xi(t) = x_3(t)$

- It is possible to write:

$$x_1(t+1) = -a_1x_1(t) - a_2x_2(t) - a_3x_3(t) + e(k), \quad x_2(t+1) = u(t) \\ u(t) = b_0e(t) + (b_1 - a_1b_0)x_1(t) + (b_2 - a_2b_0)x_2(t) + (b_3 - a_3b_0)x_3(t)$$

- The system can be written as

$$\mathbf{x}(t+1) = A_c\mathbf{x}(t) + b_ce(t), \quad u(t) = C_cu(t) + D_cu(t) \text{ where:}$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_c = [b_1 - a_1b_0 \quad b_2 - a_2b_0 \quad b_3 - a_3b_0], \quad D_c = [b_0]$$

Observer Canonical Form

- The difference equations can be written as:

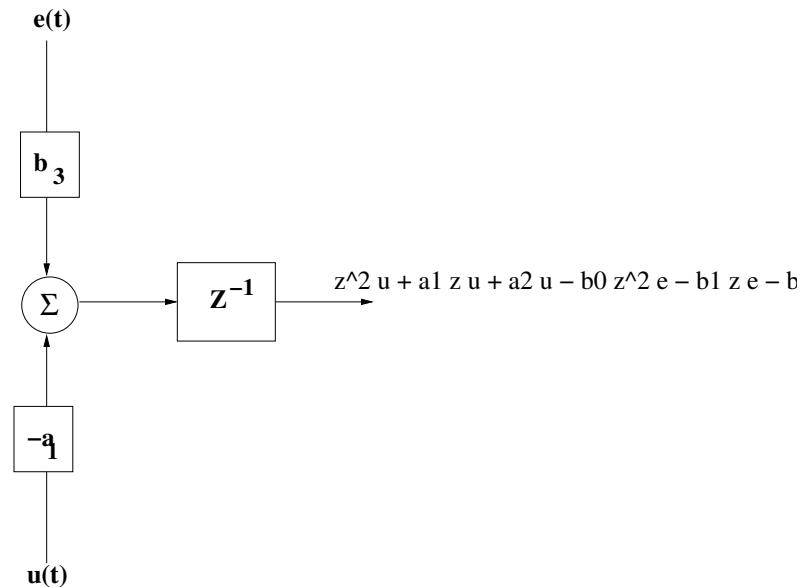
$$z^3u + a_1z^2ua_2zu + a_3u = b_0z^3e + b_1z^2e + b_2ze + b_3e$$

- We put on the lhs everything that does not depend on z :

$$b_3e - a_3u = z^3u + a_1z^2ua_2zu - b_0z^3e - b_1z^2e - b_2ze$$

Observer Canonical Form - I

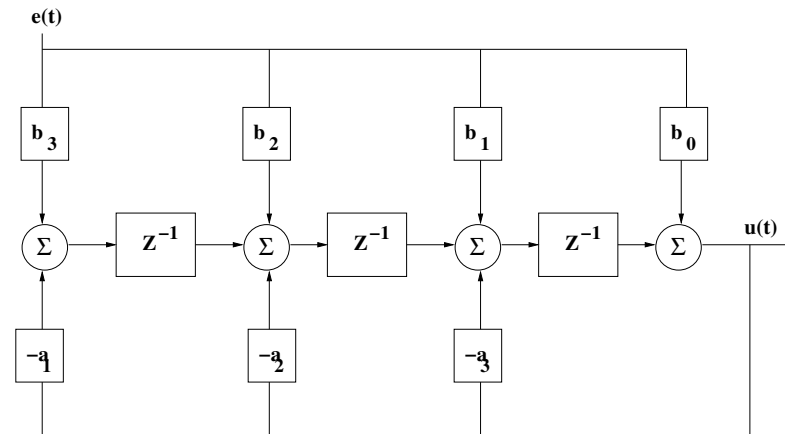
● Graphical Representation



● Hey, we can play the same game working on the output of the delay element!!

Observer Canonical Form - II

- After some iteration we get:



- The system can be written as

$\mathbf{x}(t+1) = A_o \mathbf{x}(t) + b_o e(t), u(t) = C_o \mathbf{x}(t) + D_o e(t)$ where:

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}, b_o = \begin{bmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ b_3 - b_0 a_3 \end{bmatrix}$$

$$C_o = [1 \ 0 \ 0], D_o = [b_0]$$