Ingegneria dell 'Automazione - Sistemi in Tempo Reale

Selected topics on discrete-time and sampled-data systems

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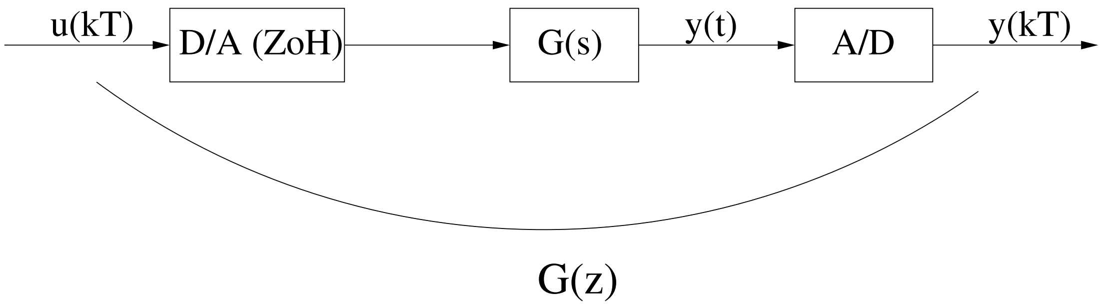
Outline

- Representation of a sampled-data system

DT models of sampled-data systems

- Using $\mathcal{Z}-$transform



- Supposed $G(s) = H(s)e^{-s\lambda}$,
- λ can be used to model computation delays
- The presence of a delay makes CT synthesis much more difficult (infinite dimension)

Example

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- $G(z) = (1 - z^{-1})\mathcal{Z}[\mathcal{L}^{-1}[e^{-lsT}(\frac{e^{msT}}{s} - \frac{e^{msT}}{s+a})] =$
  $(1 - z^{-1})z^{-l}\mathcal{Z}[\mathcal{L}^{-1}[\frac{e^{msT}}{s} - \frac{e^{msT}}{s+a}]$

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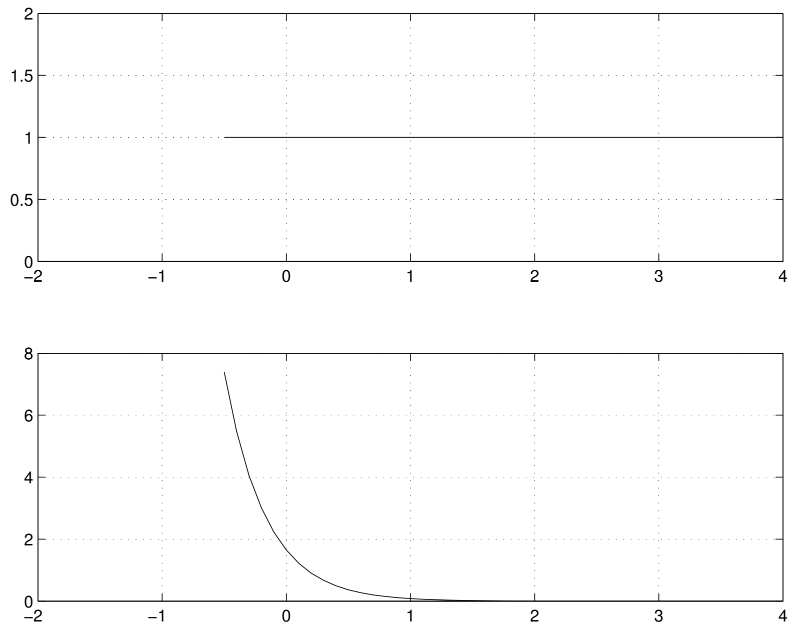

- $\mathcal{Z}[\frac{e^{smT}}{s}] = \frac{z}{z-1}$
- $\mathcal{Z}[\frac{e^{smT}}{s+a}] = \frac{ze^{-amT}}{z-e^{-aT}}$
- $G(z) = (1 - e^{-amT})\frac{z+\alpha}{z^l(z-e^{-aT})}$, where $\alpha = \frac{e^{-amT}-e^{-aT}}{1-e^{-amT}}$

Matlab Control toolbox code

```
10cm Td = 1.5
a= 1
T = 1
sysc = tf(a, [1 a],'td',Td);
sysD = c2d(sysc,T);
```

Effects of delays

- Due to the delay
  - l poles arose in the origin
  - a zero arose at $-\alpha$
    - $\alpha \rightarrow +0$ when $m \rightarrow 1$ (small delay)
    - $\alpha \rightarrow +\infty$ when $m \rightarrow 0$ (large delay)
- And now let's review some concepts.....

Zeros and poles

- Poles are the zeros of the denonimator. A pole p_i correspond to a free mode of the system associated with the time function $z = p_i^t$
- Zeros correspond to blocking properties of the the system: $z = z_i$ means that there exist initial conditions such that the system has response 0 to the signal z_i^t

Zeros and poles - I

- Example:

$$y(t+1) - ay(t) = u(t+1) - bu(t)$$

- Computing the $\mathcal{Z}-$transform we get:

$$Y(z) = \frac{z}{z-a}(y_0 - u_0) + \frac{z-b}{z-a}U(z)$$

- $u(t) = b^t \Rightarrow U(z) = \frac{z}{z-b}$
- Therefore

$$Y(z) = \frac{z}{z-a}(y_0 - 1)$$

- if $y_0 = 1$ then $Y(z) = 0$

Zeros and poles - II

- A more subtle example:

$$y(t+1) - ay(t) = u(t+1) - au(t)$$

The transfer function is $\frac{z-a}{z-a}$; is it equivalent to 1?

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- Problems are limited only if $|a| < 1$

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- $U(z)$ can be expanded as:
  $U(z) = 1 + \frac{a(a-b)}{(a-c)(z-a)} - \frac{c(c-b)}{(a-c)(z-c)}$

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- $u(t)$ grows unbounded: exact tracking with finite commands is impossible

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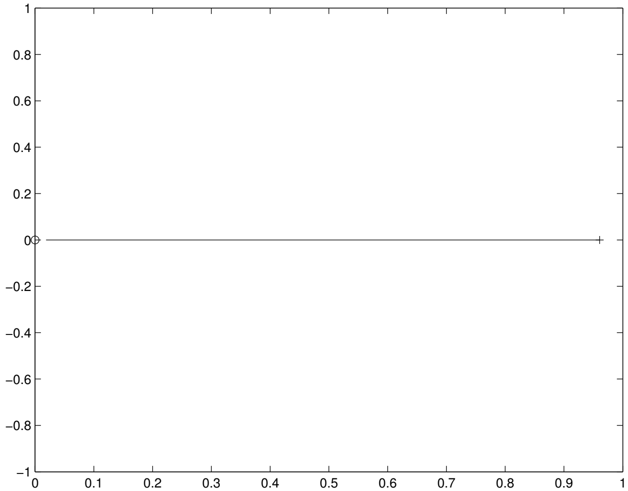

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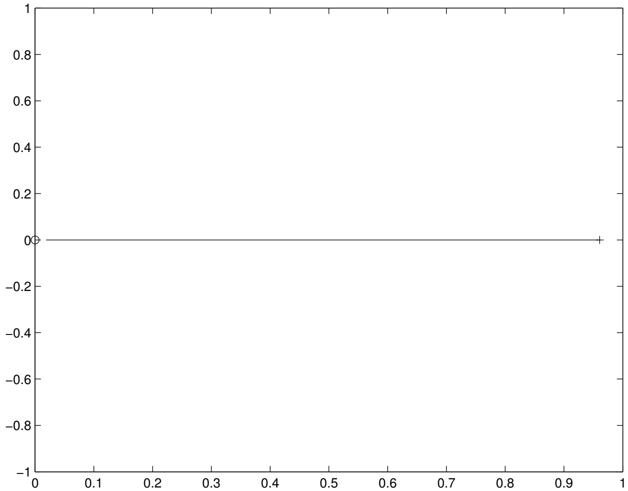

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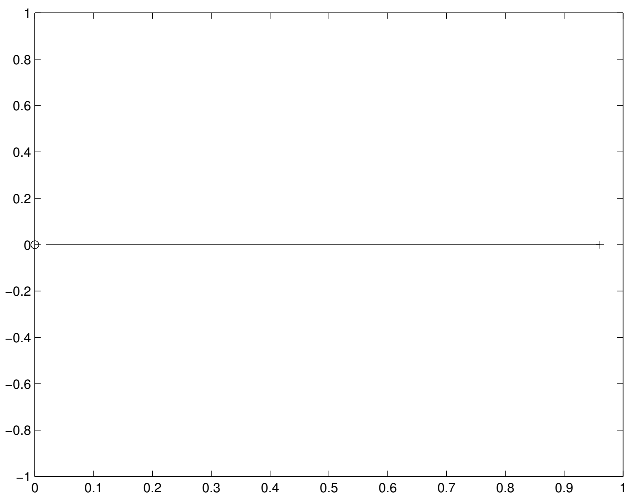

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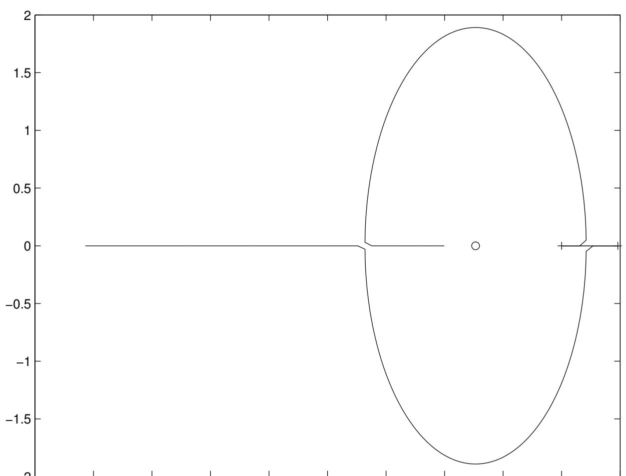

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- Jury criterion!

Jury stability criterion

- Consider the stability of $A(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$
- Write the following table

$$
\begin{array}{ccccccc}
a_0 & a_1 & \ldots & a_{n-1} & a_n & & \\
a_n & a_{n-1} & \ldots & a_1 & a_0 & & \alpha_n = \frac{a_n}{a_0} \\
\hline
a_0^{n-1} & a_1^{n-1} & \ldots & a_{n-1}^{n-1} & & & \\
a_{n-1}^{n-1} & a_{n-2}^{n-1} & \ldots & a_0^{n-1} & & \alpha_{n-1} = \frac{a_{n-1}^{n-1}}{a_0^{n-1}} & \\
\hline
\ldots & & & & & & \\
a_0^0 & & & & & &
\end{array}
$$

where $a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k$, $\alpha_k = a_k^k / a_0^k$

The Jury-Criterion

Theorem 0 (Jury 1961) *If* $a_0 > 0$ *then the system has all roots inside the unit circle if and only if all* a_0^k *with* $k = 0, 1, \ldots, n-1$ *are positive. If no* a_0^k *is zero, then the number of negative* a_0^k *is equal to the number of roots outside the unit circle.*

The two-dimensional case

- Consider the stability of $A(z) = z^2 + a_1 z + a_2$
- Jurys table is

$$
\begin{array}{cccc}
1 & a_1 & a_2 & \\
a_2 & a_1 & 1 & \alpha_2 = a_2 \\
\hline
 & & & \\
1 - a_2^2 & a_1(1-a_2) & & \\
a_1(1-a_2) & 1 - a_2^2 & & \alpha_1 = \frac{a_1}{a_2+1} \\
\hline
 & & & \\
1 - a_2^2 - a_1^2 \frac{1-a_2}{1+a_2} & & &
\end{array}
$$

- Imposing conditions in the theorem one gets:

$$
\begin{aligned}
a_2 &< 1 \\
a_2 &> -1 + a_1 \\
a_2 &> -1 - a_1
\end{aligned}
$$

Going back to our example

- The equation is $z^2 + (k\omega - e^{-aT})z + k\omega\alpha$, where $\omega = 1 - e^{-amT}$, $\alpha = \frac{e^{-amT} - e^{-aT}}{\omega}$
- applying the Jury criterion, the system is stable iff

$$\begin{array}{ll} -\frac{e^{-aT}+1}{2e^{-amT}-e^{-aT}-1} < k < \frac{1}{e^{-amT}-e^{-aT}} & 0 < m < -\frac{\log \frac{e^{-aT}-1}{2}}{aT} \\ -1 < k < \min\{\frac{1}{e^{-amT}-e^{-aT}}, -\frac{1+e^{-aT}}{2e^{-amT}-e^{aT}-1}\} & \text{otherwise} \end{array}$$

- The stabiliy margin decreases with the introduced dealy

DT models of sampled-data systems

Using state space representation

- Consider a system $\dot{x} = Fx + Gu, y = Hx$
- The ZoH equivalent is given by:

$$x((k+1)T) = \Phi x(k) + \Gamma u(k)$$
$$y = Cx(k)$$

where $\Phi = e^{FT}$, $\Gamma = \int_0^T e^{F\tau} d\tau G$

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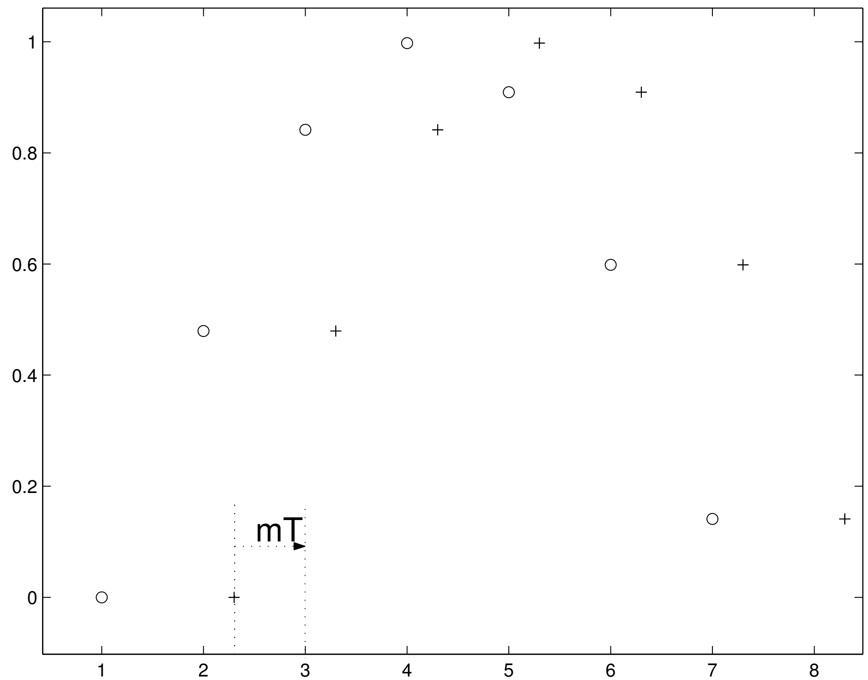
- $\Phi = e^{FT} = I + FT + \ldots = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$
- $\Gamma = [IT + FT^2/2! + F^2T^3/3! + \ldots]G = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$

What about delay?

- Consider the system: $\dot{x} = Fx + Gu(t-\lambda),\ \ y = Hx$
- The general solution is given by
  $x((k+1)T) = e^{FT}x(kT) + \int_{kT}^{(k+1)T} e^{F((k+1)T-\tau}Gu(\tau-\lambda)d\tau$
- Let $\lambda = lT - mT$, with $l \in \mathbb{N}$, $m \in [0,\ 1)$



Delays

- We come up with
  $x((k+1)T) = e^{FT}x(kT) + \Gamma_1 u((k-l)T) + \Gamma_2 u((k-l+1)T)$
- where: $\Gamma_1 = \int_{kT}^{kT+(1-m)T} e^{F(k+1)T-\tau} G d\tau$,
  $\Gamma_2 = \int_{kT+(1-m)T}^{(k+1)T} e^{F(k+1)T-\tau} G d\tau$
- with the change of variable: $\eta = (k+1)T - \tau$ we get:

$$\Gamma_1 = \int_{mT}^{T} e^{F\eta} G d\eta$$
$$\Gamma_2 = \int_{0}^{mT} e^{F\eta} G d\eta$$

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- The system dynamic equation becomes:

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Gamma_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} \Gamma_2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [H,\ 0]\left[\begin{smallmatrix} x(k) \\ z(k) \end{smallmatrix}\right]$$

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- New state variables

$$z_1(k) = u(k-l)$$
$$z_2(k) = u(k-l+1)$$
$$\dots$$
$$z_l(k) = u(k-1)$$

The case $l > 1$ - I

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$$y(k) = [H,\, 0,\, 0\, \dots,\, 0] \begin{bmatrix} x(k) \\ z_1(k) \\ \dots \\ z_l(k) \end{bmatrix}$$

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- The system with $l = 0, m \neq 0$ will show the same response at $t = 0$ that the origal system would show at $t = mT$
- This is a tool to study the intersample behaviour (invented by Jury and originally called modified $\mathcal{Z}-$transform

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- After some computation we get:

$$\xi(k+1) = \Phi \xi(k) + \Gamma u(k)$$
$$y(k) = H\xi(k) + Ju(k)$$

where $\Gamma = (\Phi\Gamma_2 + \Gamma_1)$, $J = H\Gamma_2$

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- Augmented discrete-time system:

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} e^{fT} & \frac{e^{fT} - e^{fmT}}{f} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} \frac{e^{mfT} - 1}{f} \\ 1 \end{bmatrix} u(k)$$

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- Control by static feedback $u(k) = kx(x)$ yields the closed loop dynamic:

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} e^{fT} + k\frac{e^{mfT} - 1}{f} & \frac{e^{fT} - e^{fmT}}{f} \\ k & 0 \end{bmatrix}$$

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- Stability can be studied- via Jury criterion - on

$$z^2 - (e^{fT} + k\frac{e^{mfT} - 1}{f})z - k\frac{e^{fT} - e^{fmT}}{f}$$

Summing up....

- Delays are relatively easy to model
- They extend the model with addtional state variables
- This effect has to be carefully accounted for in control design