Laurea Specialistica in Ingegneria dell'Automazione

Sistemi in Tempo Reale

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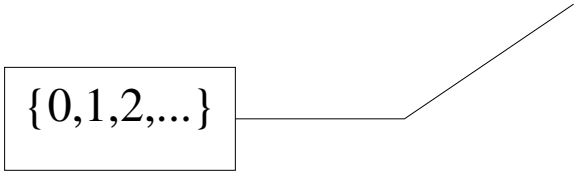
FSM

Outline

- Finite State Machines (FSMs)

The concept of SM

- SM are systems operating on sequences:
  - *F: InputSignals -> OuputSignals*
    where both input signals and output signals have the form:
    - EventStream: $Naturals_0$ -> Symbols



  - This definition captures the concept of *ordering* between events
  - Reference to time (discrete or continuous) is neither explicit nor implied
- Crucial is the definition of a *state* that summarises the past story of the system

Example

$$x\in InputSignals=[Nats_0\to\{0,1\}]$$

$$y\in OutputSignals=[Nats_0\to\{True,False\}]$$

$$Recognizer(x)(n)=\begin{cases} True & if\ x(n)=0\wedge x(n-1)=1\wedge x(n-2)=0 \\ False & otherwise \end{cases}$$

Example:

$$x=\{0,0,0,1,0,0..\}\to y=\{-,-,False,False,True,False..\}$$

A formal definition

A state machine has several components:

States Set of possible states (called state space)

Inputs Set of possible input elements (called input alphabet)

Outputs Set of possible output elements (called output alphabet)

update: States x Inputs -> States x Outputs the update function

The update function defines the new state and output given the current state and input.

initialState The initial state

Thus, a state machine can be described as a 5-tuple:

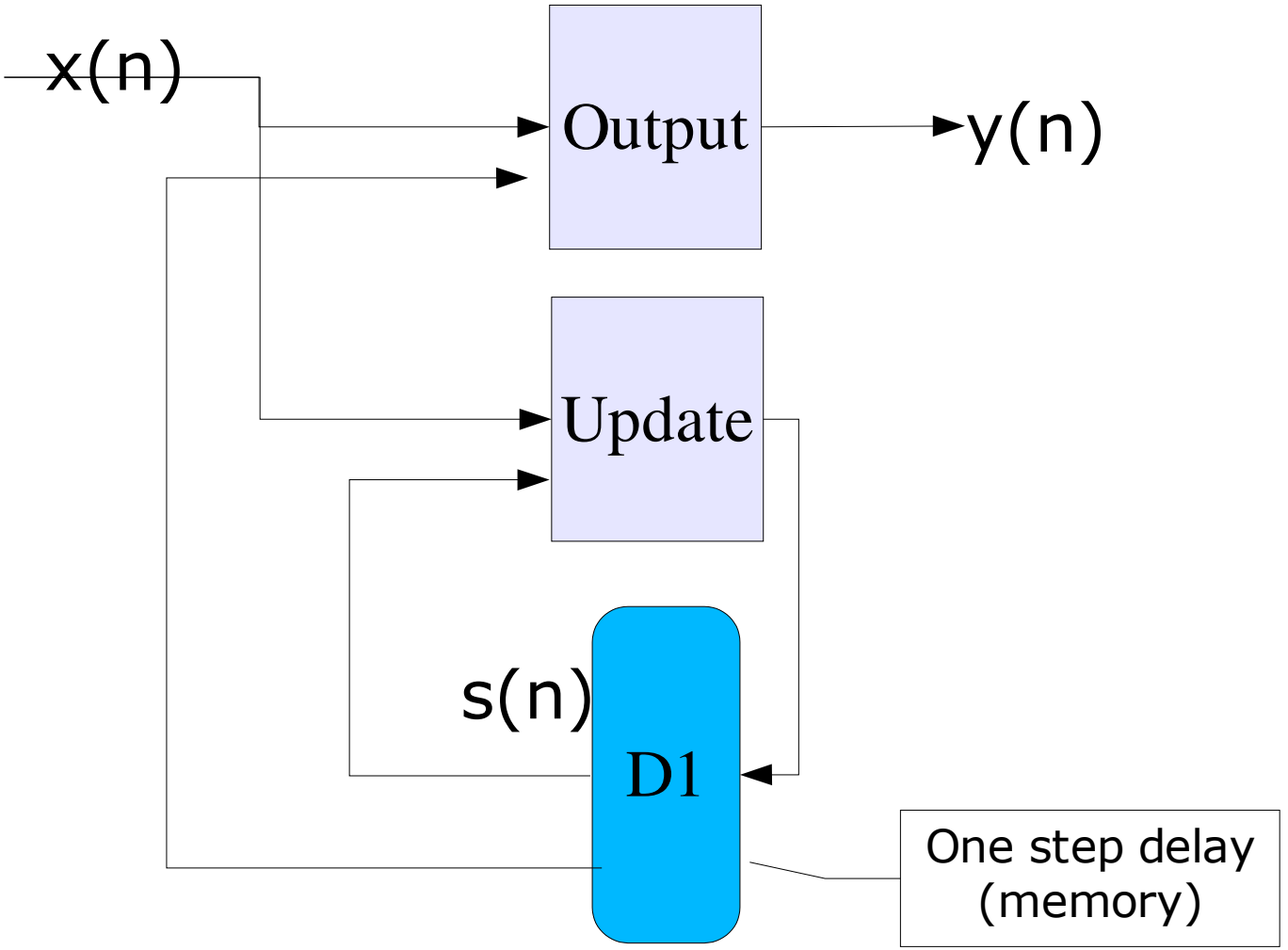
(*States, Inputs, Outputs, update, initialState*)

Update

- SM are causal: output depends only on the current state (that summarises the past history of the system) and on the current inputs
- The update dunction is often split into two functions:

$$nextState : States \times Inputs \rightarrow States, \quad s(n+1) = nextState(s(n), x(n))$$
$$output : States \times inputs \rightarrow outputs \quad y(n) = output(s(n), x(n))$$

Conceptual scheme



Stuttering

We define a special “do nothing” input called *absent*.

When x(n) = *absent*,

- the state does not change, so s(n+1) = s(n);
- the output is also “nothing”, i. e. y(n) = *absent*.
- We can always add absent symbols to input sequences without changing non absent outputs

This symbol, *absent*, is called the stuttering symbol.

Note that this symbol is always a possible input and output, so

$$absent \in Inputs\ ,\ absent \in Outputs$$

Example

The Parity System

$$\text{Parity} : [\ \text{Nats}_0 \rightarrow \text{Bools}\ ] \rightarrow [\ \text{Nats}_0 \rightarrow \text{Bools}\ ]$$

such that $\forall\ x \in [\ \text{Nats}_0 \rightarrow \text{Bools}\ ]\ ,\ \forall\ y \in \text{Nats}_0\ ,$

$$(\ \text{Parity}\ (x)\ )\ (y) = \begin{cases} \text{true} & \text{if}\ \ |\ \text{trueValues}\ (x,y)\ |\ \ \text{is even} \\ \text{false} & \text{if}\ \ |\ \text{trueValues}\ (x,y)\ |\ \ \text{is odd} \end{cases}$$

where $\text{trueValues}\ (x,y) = \{\ z \in \text{Nats}_0\ |\ z < y \wedge x\ (z) = \text{true}\ \}$

Example - (continued)

State-Machine Implementation of the Parity System

State after i-th input :

true, if first i inputs contain even number of true's

false, if first i inputs contain odd number of true's

Two states

Example - (continued)

Inputs = Bools, Outputs = Bools

States = Bools

initialState = true

nextState : $States \times Inputs \rightarrow States$

such that $\forall\, q \in States,\ \forall\, x \in Inputs,$

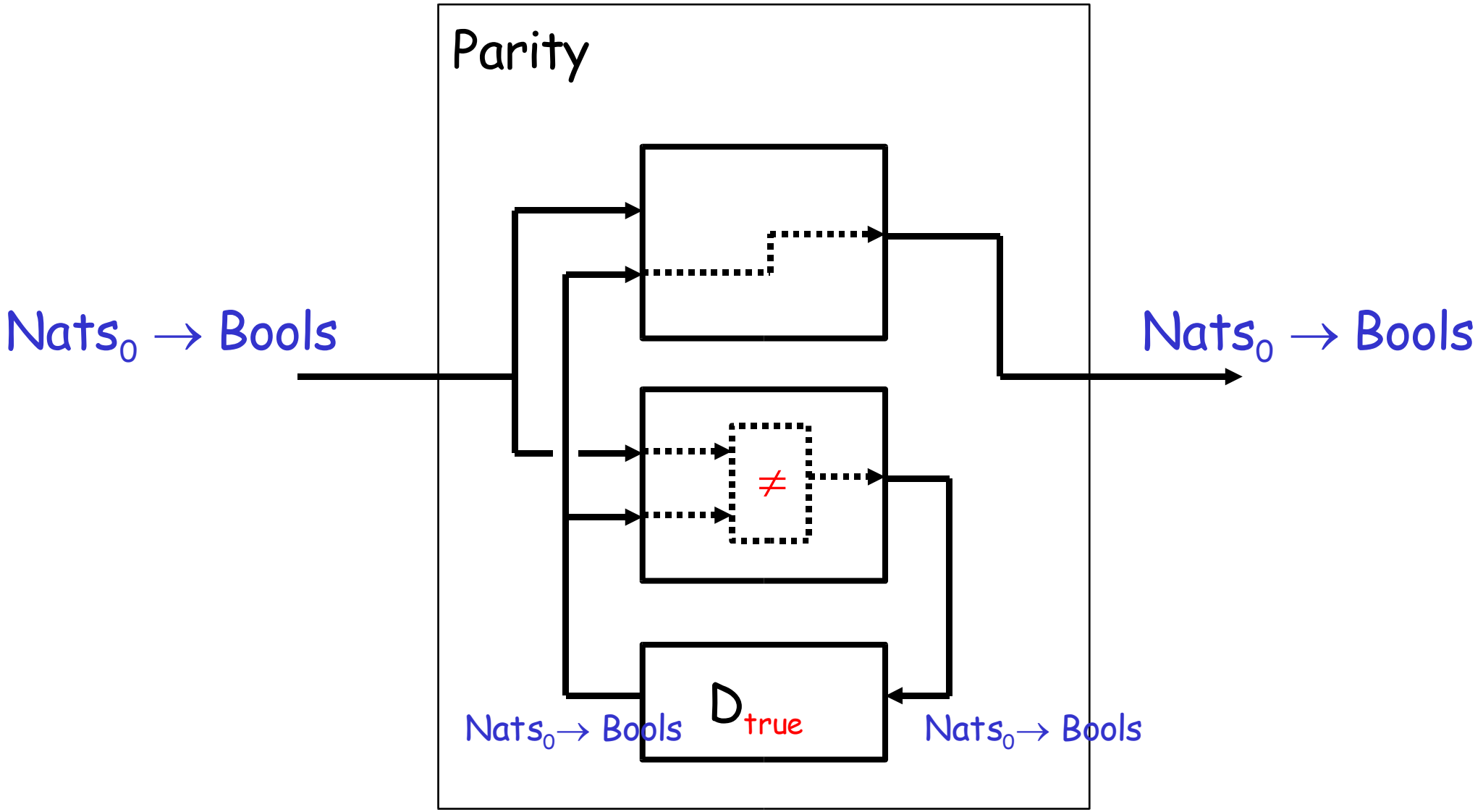
$nextState(q,x) = (q \neq x)$

output : $States \times Inputs \rightarrow Outputs$

such that $\forall\, q \in States,\ \forall\, x \in Inputs,$

$output(q,x) = q$

Example - (continued)



FSM

- Both examples shown above have a common features:
  A finite number of states
- For this reason they are called Finite State Machines
- We will also restrict to the case of finite input and output alphabet

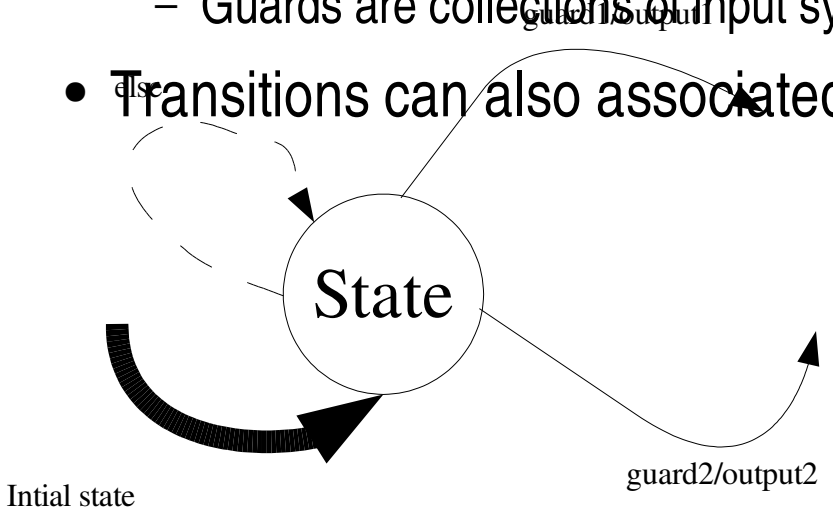
Update tables

- Update functions can be specificied using tables
- Parity example:

	(s(n+1), y(n)) = *update*(s(n), x(n))	
	x(n) = *True*	x(n) = *False*
s(n) = True	(False, True)	(True, True)
s(n) = False	(True, False)	(False, False)

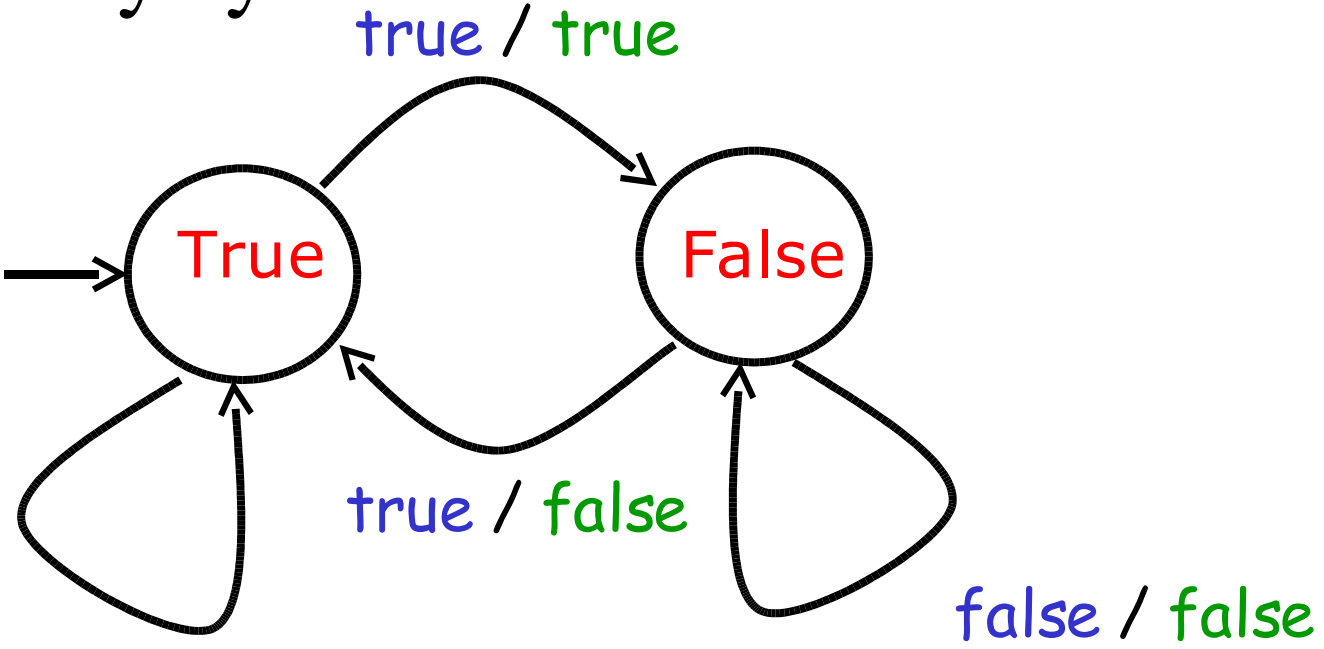
State transition diagrams

- State transition diagrams are a popular way for representing state machines
  - Bubbles represent states
  - Directed Arcs represent transitions
    - Transition are enabled by a guard
      - Guards are collections of input symbols
    - Transitions can also associated to outputs



Example

The parity system



States = Bools

Inputs = Bools

Outputs = Bools

Shorthand

- If no guard is specified the transition is always taken (except for the stuttering symbol)
- If no output symbol is specified then the output is the stuttering symbol
- If no else transition is specified then we assume a self loop
- If inputs are {a,b,c,d,...,z} we can use {not a} in a guard instead of {b,c,...,z}

Excercise

Draw the transition diagram of a state machine with

Inputs = Bools

Outputs = Bools

At all times t, the output is true iff the inputs at times t-2, t-1, and t are all true .

Exercise - (continued)

State after i-th input :

0, if i-th input is false (or i = 0)

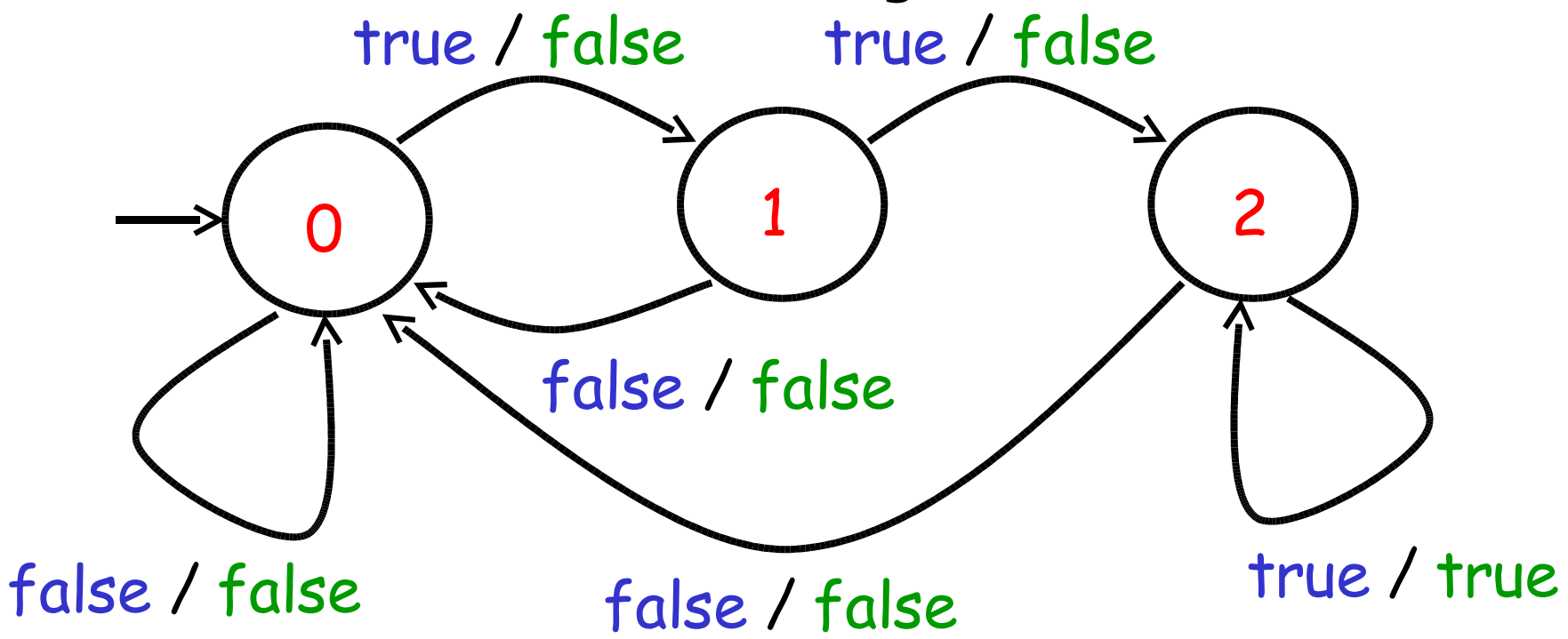
1, if i-th input is true and (i-1)-st input is false (or i = 1)

2, if both i-th and (i-1)-st inputs are true

Three states

Exercise - (Continued)

Transition Diagram



States = { 0, 1, 2 }

Inputs = Bools

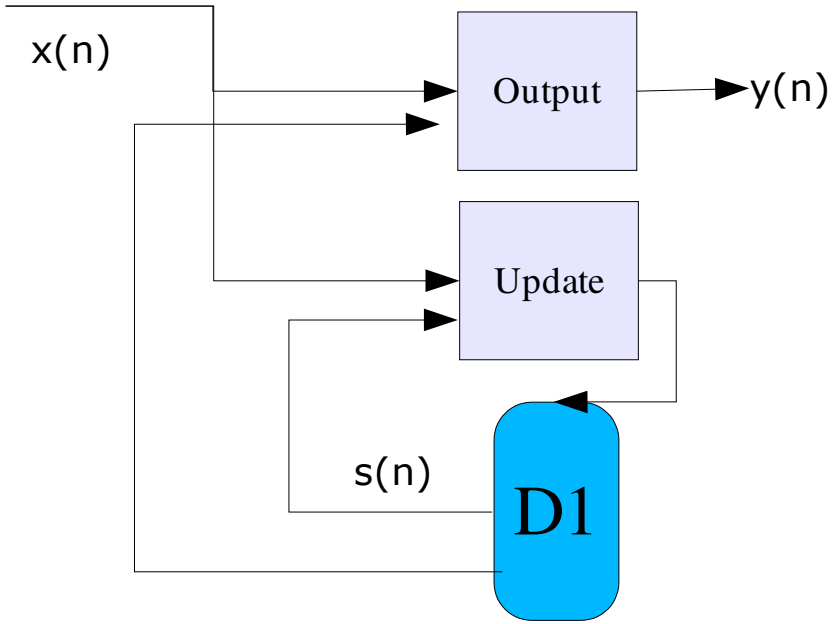
Outputs = Bools

Some terminology

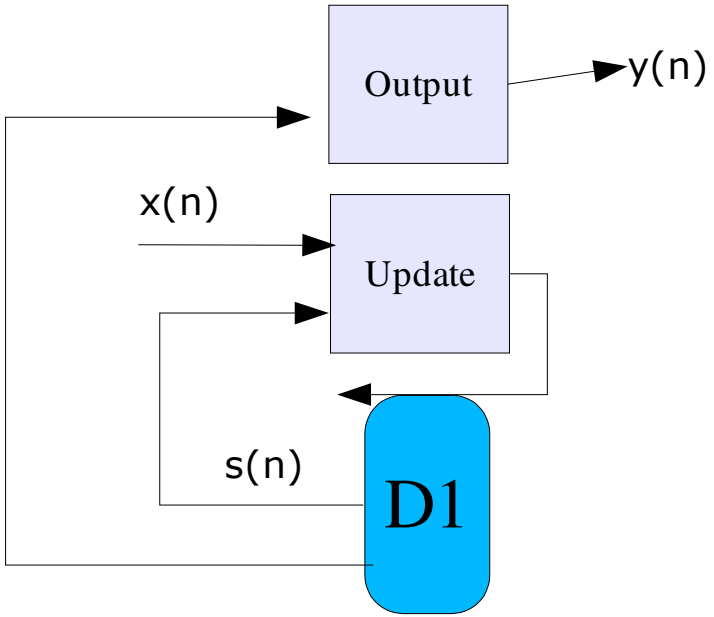
- **Receptiveness:** our SM always react to a symbol (there is an outgoing arc specified, or an else arc either specified or implied)
- **Determinism:** a SM is said deterministic if every input symbols is contained in one and only one outgoing arc
- State transitions triggered by a sequence of inputs are called **State response**
- A finite machine without output is called *finite automata*

Some definitions - (continued)

- The state machines addressed here are called Mealy machines. Mealy machines generate outputs during state transitions.
- Moore machines generate output while the system is in a particular state (output depends on state only).



Mealy



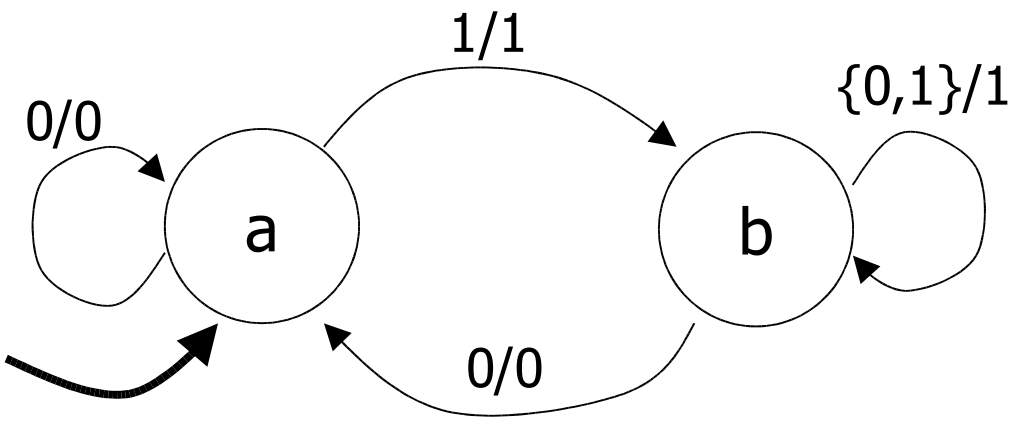
Moore

Some facts

- **A SM is state-determined:** Each transition and output depends only on the current state and current input.
  - Previous input elements only affect the transitions and output insofar as they determine the current state.
- A representation of a SM may not be unique
- Representations of a SM
  - Update tables (useful in computer implementations)
  - State transition diagrams (human friendly)
  - Sets and Functions (useful for mathematical purposes)

Nondeterministic State Machines

Nondeterministic state machines are like the deterministic state machines, except there may be more than one possible transition for a given current state and input.



When in state s(n)=b,
if x(n) = 0,
the next state s(n+1)
can be either a or b.

The output y(n)
can be either 0 or 1.

In a nondeterministic state machine, there is more than one possible state response and output sequence.

Nodeterministic State Machines: Parts

Nondeterministic state machines are described with the 5-tuple:

(*States, Inputs, Outputs,* ***possibleUpdates,*** *initialState*)

The update function is now called ***possibleUpdates***.

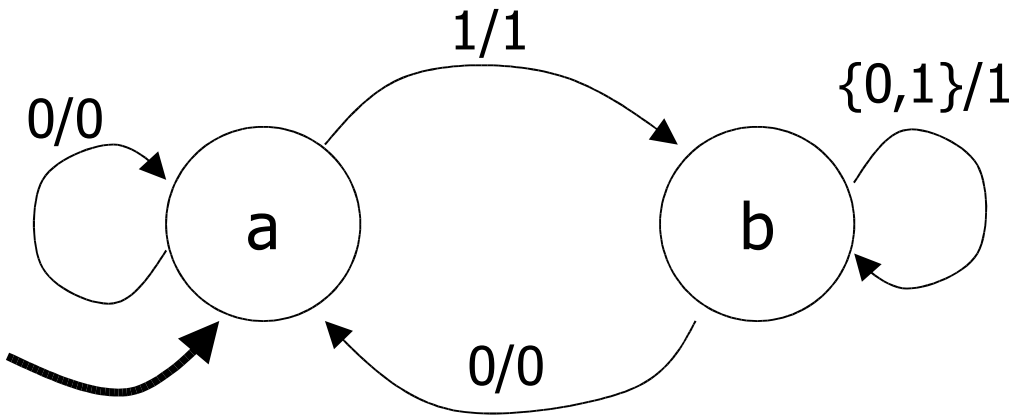
For a given input x(n) and current state s(n), *possibleUpdates* provides the **set** of possible next states s(n+1) and outputs y(n).

possibleUpdates: *States* × *Inputs* -> P(*States* × *Outputs*)

Here, the notation P() denotes the power set. For some set S, P(S) is the set of all subsets of S.

Nondeterministic State Machines: Example

Provide the 5-tuple definition for the following state machine:



States = {a, b}

Inputs = {0, 1, absent}

Outputs = {0, 1, absent}

initialState = a

	(s(n+1), y(n)) = *possibleUpdates*(s(n), x(n))	
	x(n) = 0	x(n) = 1
s(n) = a	(a, 0)	(b, 1)
s(n) = b	{(b, 1), (a, 0)}	(b, 1)

Why nondeterminism?

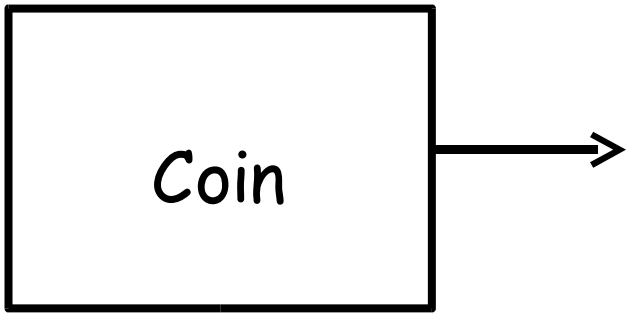
-modeling randomness

-modeling abstraction

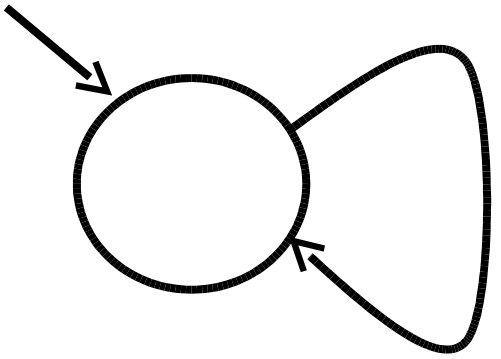
-...

Modeling randonmess

Coin Tossing



$Nats_0 \rightarrow \{ heads, tails \}$



toss / heads

toss / tails

Modeling randonmess

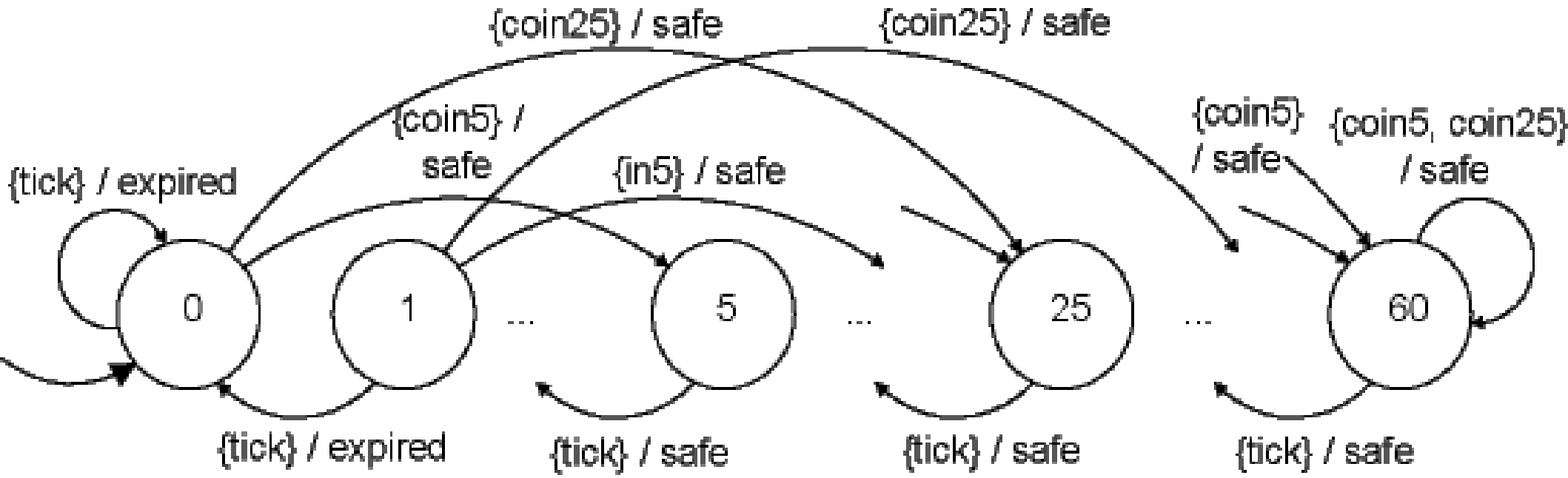
One possible behavior

Time	0	1	2	3	4
Input		toss	toss	toss	toss
Output	heads	heads	tails	heads	tails

Another possible behavior

Time	0	1	2	3	4
Input		toss	toss	toss	toss
Output	tails	heads	tails	heads	heads

Nondeterministic State Machines: Abstraction



This example is a deterministic model of a parking meter.

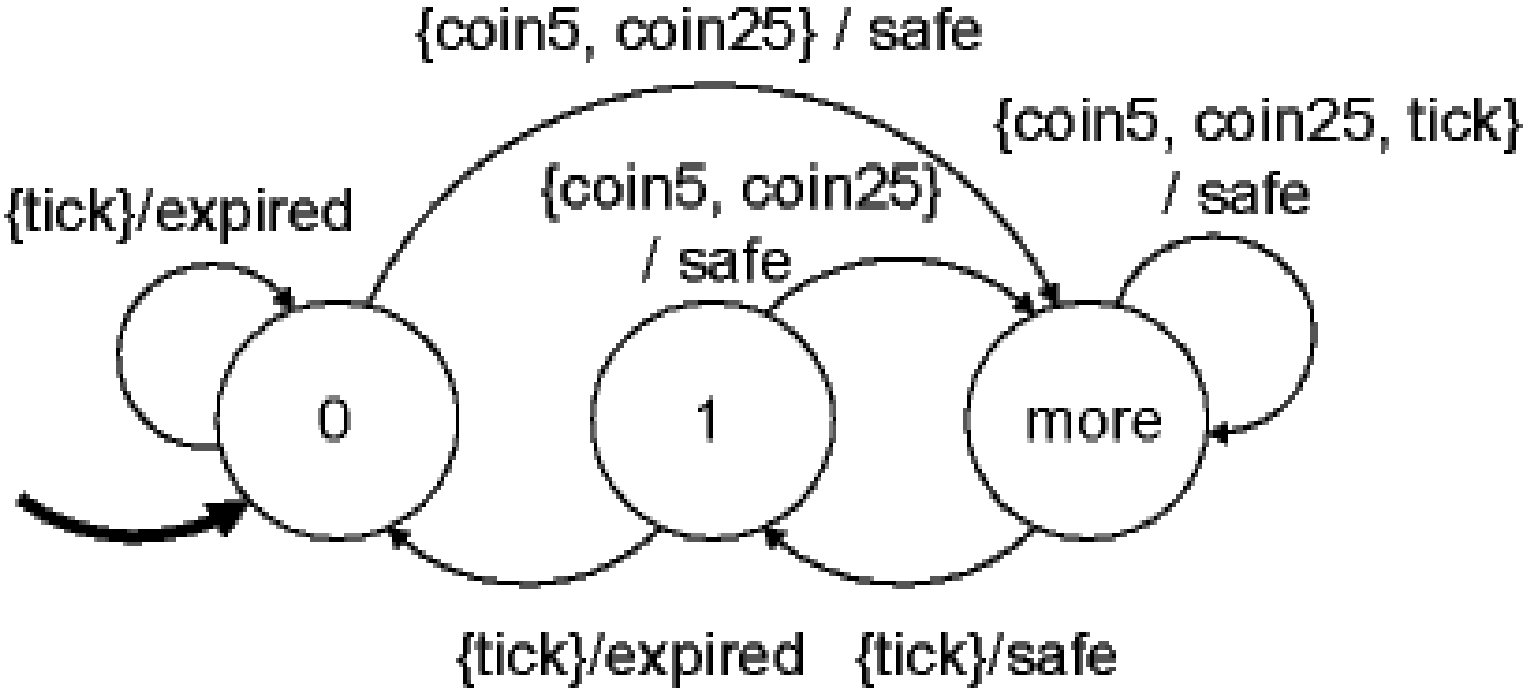
The inputs are coin5 (5 minutes) coin25 (25 minutes) and tick.

The outputs are safe and expired.

The states represent the number of minutes left until expiration.

This model of meter operation is accurate, but complicated!

Nondeterministic State Machines: Abstraction



Here is a nondeterministic model of the parking meter.

It contains less detail – it is an abstraction of the previous model.

But, some outputs are possible in this model that are impossible in the previous model:

x = coin5, tick, tick can result in y = safe, safe, expired

The meter expires 2 minutes after inserting a 5 minute coin!

State Machines: Behaviors

A behavior of a state machine is a pair (x, y) where x is an input sequence and y is a corresponding output sequence:

Behaviors = {(x, y) $\in$ [*Naturals*$_0$ ->*Inputs*] $\times$ [*Naturals*$_0$ ->*Outputs*] | y is a possible output sequence for the input x}

Deterministic behaviours

Deterministic SM:

for every input signal, there is exactly one output signal.

Behavoiurs is the graph of a Function:

$$DetSys : [\ Nats \rightarrow Inputs\ ] \rightarrow [\ Nats \rightarrow Outputs\ ]$$

Nondeterministic behaviours

Nondeterministic Reactive System:

for every input signal, there is one or more output signals.

Behaviours are a Binary relation:

$$NondetSys \subseteq [\, Nats \to Inputs \,] \times [\, Nats \to Outputs \,]$$

such that $\forall\, x \in [\, Nats \to Inputs \,]$,
$\exists\, y \in [\, Nats \to Outputs \,],\ (x,y) \in NondetSys$

Every pair $(x,y) \in NondetSys$ is called a behavior.

Refinement

S1 is a more detailed description of S2;

S2 is an abstraction or property of S1.

System S1 refines system S2

iff

1. Time [S1] = Time [S2] ,
2. Inputs [S1] = Inputs [S2] ,
3. Outputs [S1] = Outputs [S2] ,
4. Behaviors [S1] $\subseteq$ Behaviors [S2] .

Equivalence

Systems S1 and S2 are equivalent

iff

1. Time [S1] = Time [S2] ,
2. Inputs [S1] = Inputs [S2] ,
3. Outputs [S1] = Outputs [S2] ,
4. Behaviors [S1] = Behaviors [S2] .

Simulation

A binary relation $S \subseteq States[M1] \times States[M2]$ is a simulation of M1 by M2 iff

1. $\forall p \in possibleInitialStates[M1]$,

   $\exists q \in possibleInitialStates[M2], (p, q) \in S$ and

2. $\forall p \in States[M1], \forall q \in States[M2]$,

   if $(p, q) \in S$,

   then $\forall x \in Inputs, \forall y \in Outputs, \forall p' \in States[M1]$,

   if $(p', y) \in possibleUpdates[M1](p, x)$

   then $\exists q' \in States[M2]$,

   $(q', y) \in possibleUpdates[M2](q, x)$ and

   $(p', q') \in S$.

Conditions for refinement

condition on behaviors

Theorem :

The nondeterministic state machines M1 refines the nondeterministic state machine M2

if

there exists a simulation of M1 by M2 .

relation between states

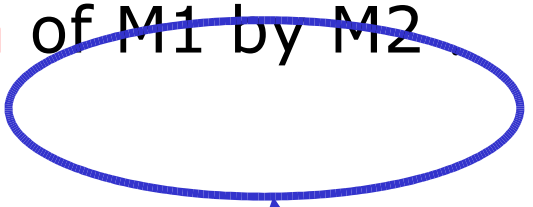
Beware

Theorem :

The nondeterministic state machines M1 refines the nondeterministic state machine M2

if

there exists a simulation of M1 by M2



Not "if-and-only-if" !

Not symmetric !

(We say that "M2 simulates M1".)

Intuition

- The theorem simply states that M2 can match every move of m1 producing the same output
- Moreover, if M2 cannot produce an output sequence, neither can M1, as stated below:

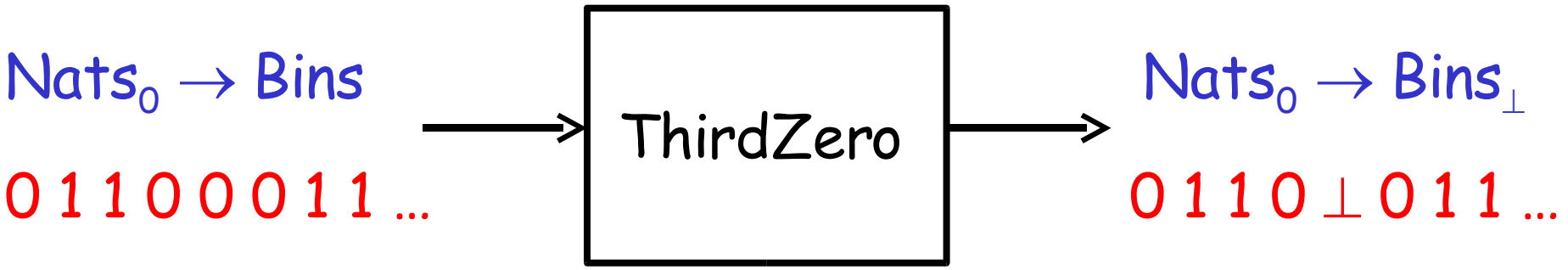
Corollary:

Let M2 simulate M1. If $(x,y) \notin Behaviours_{M2}$ then

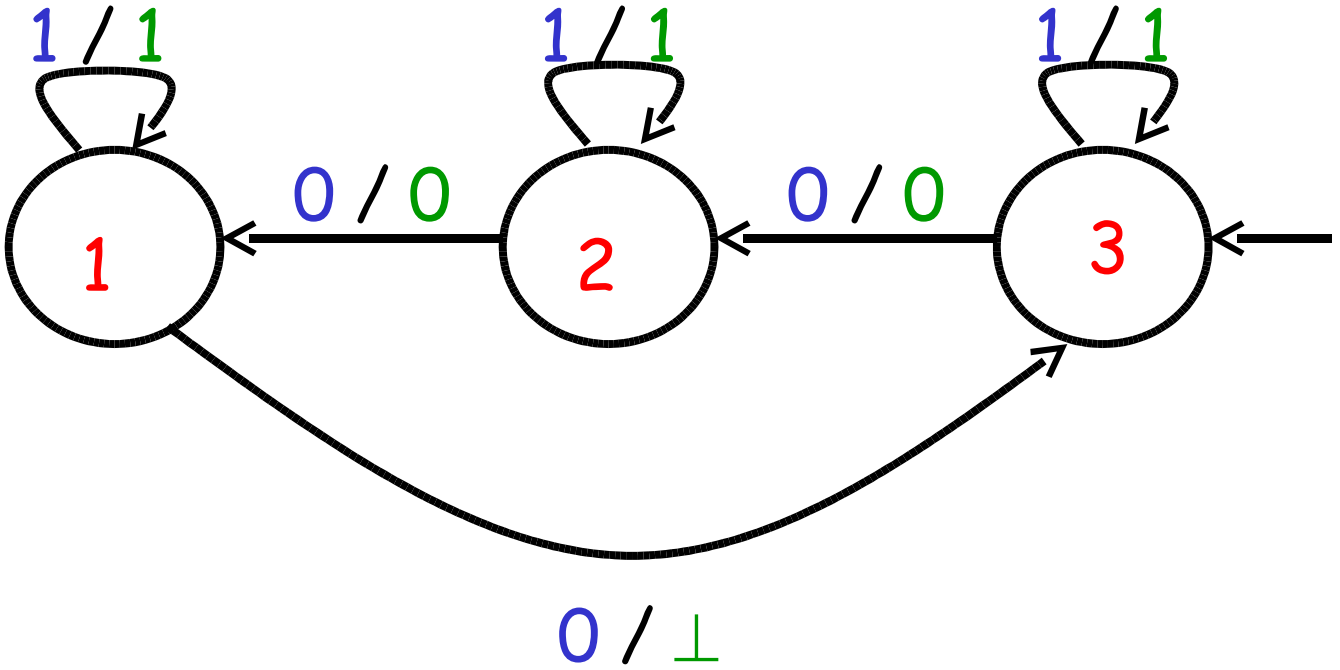
$(x,y) \notin Behaviours_{M1}$.

Example

Channel that drops every third zero



Example - (continued)



State between time t-1 and time t:

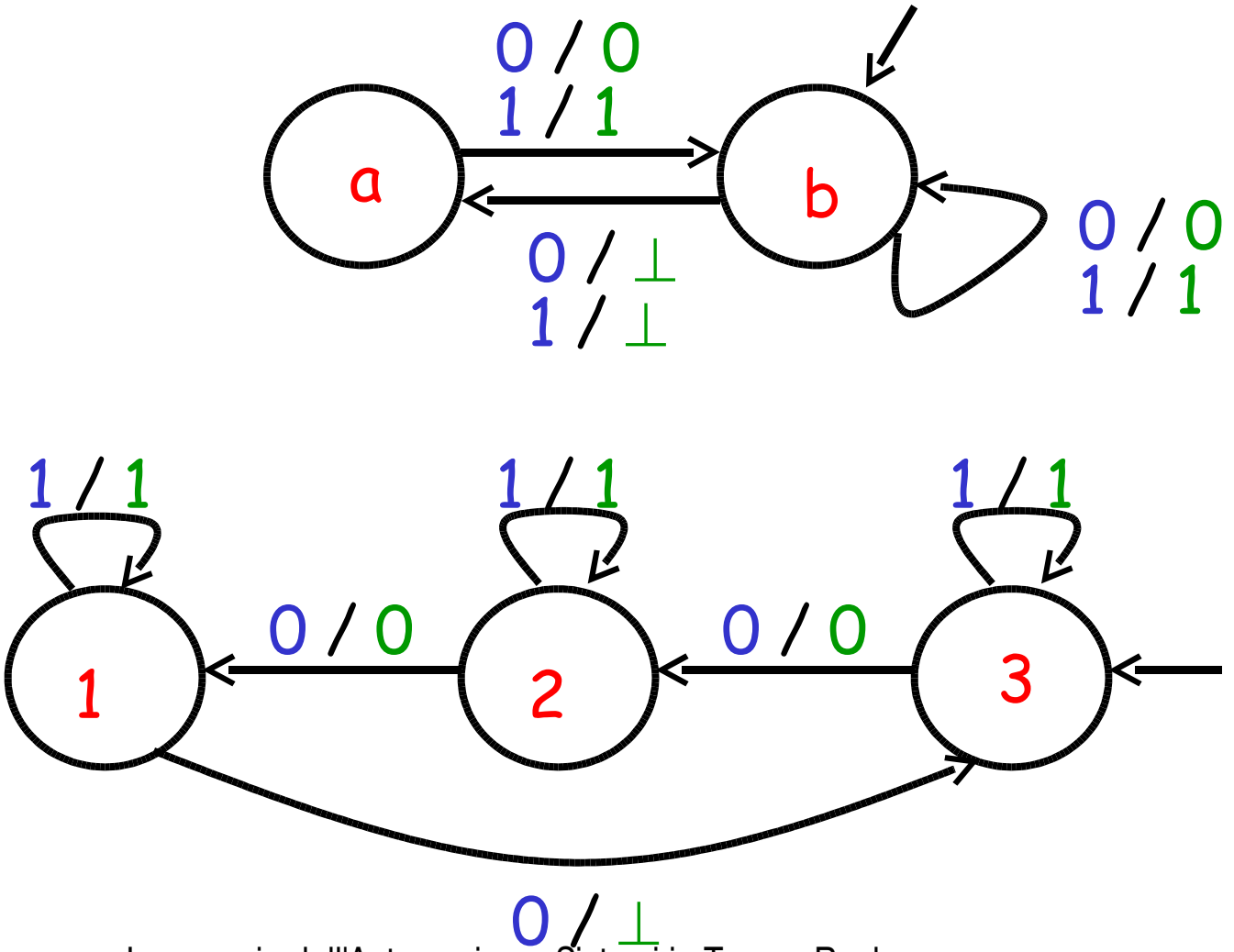
3 third zero from now will be dropped

2 second zero from now will be dropped

1 next zero will be dropped

Example - (continued)

ThirdZero refines NotTwice

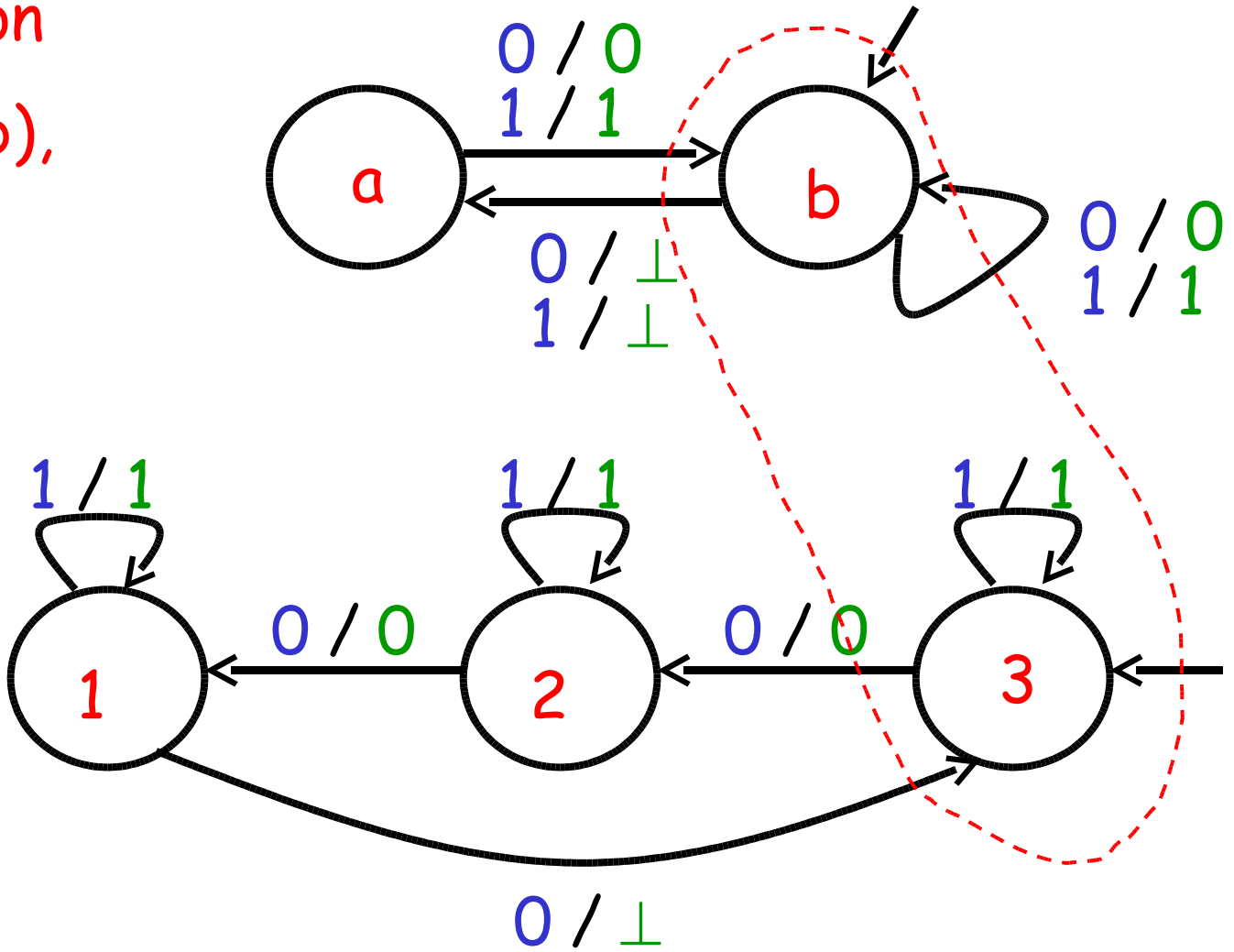


Example - (continued)

ThirdZero is simulated by NotTwice

Simulation

S = { (3,b),

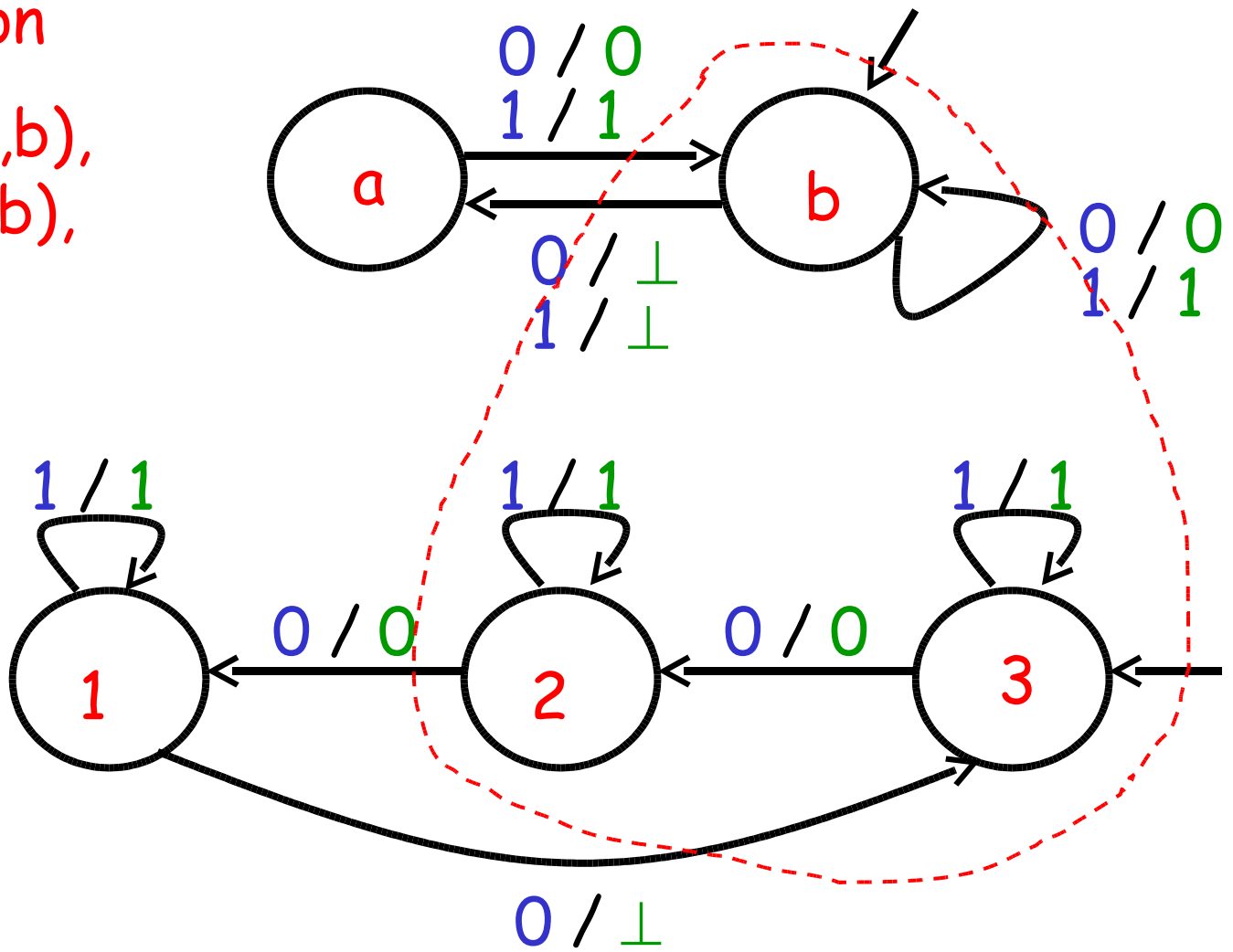


Example - (continued)

ThirdZero is simulated by NotTwice

Simulation

S = { (3,b),
(2,b),

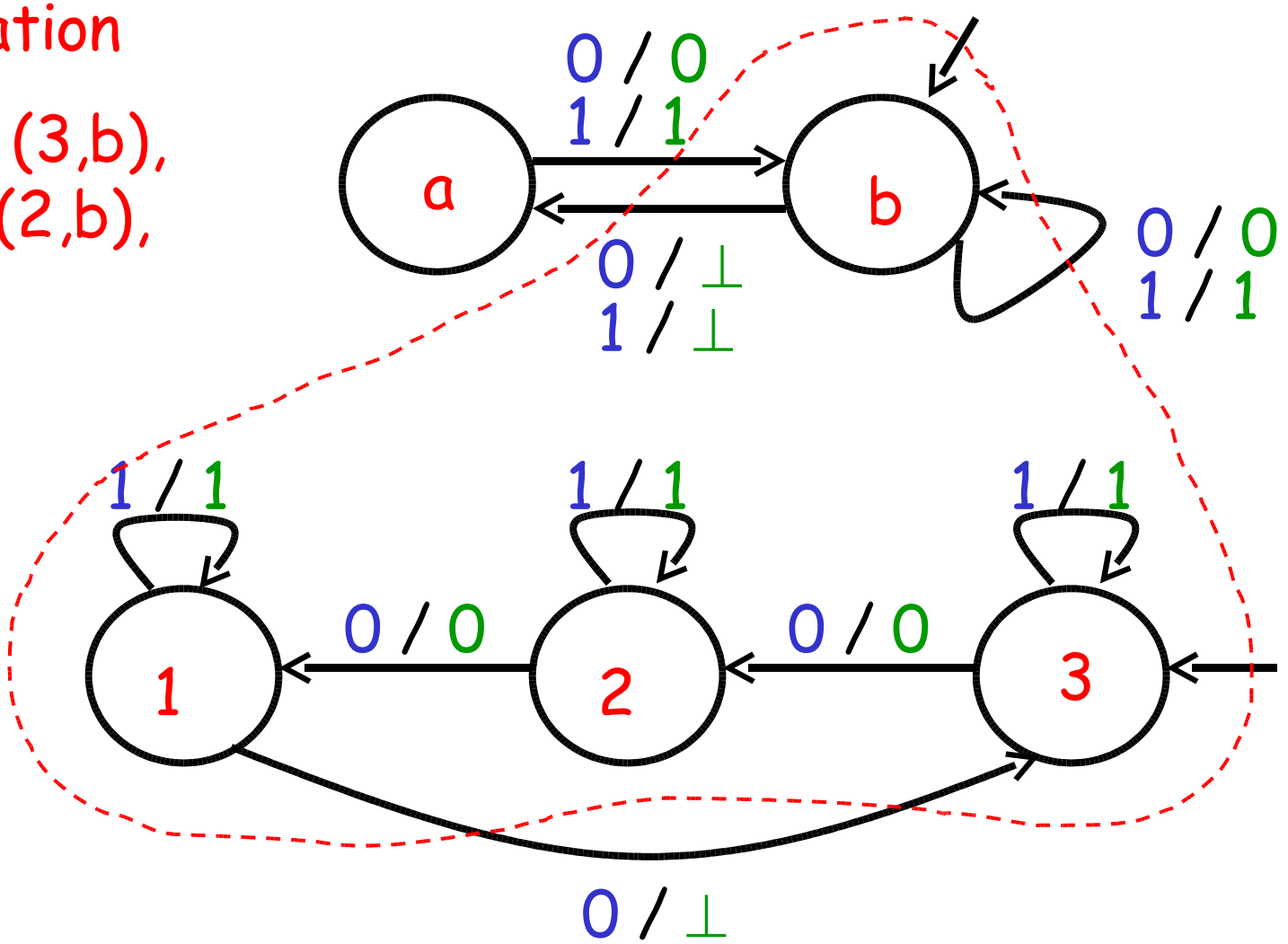


Example - (continued)

ThirdZero is simulated by NotTwice

Simulation

S = { (3,b),
(2,b),
(1,b),

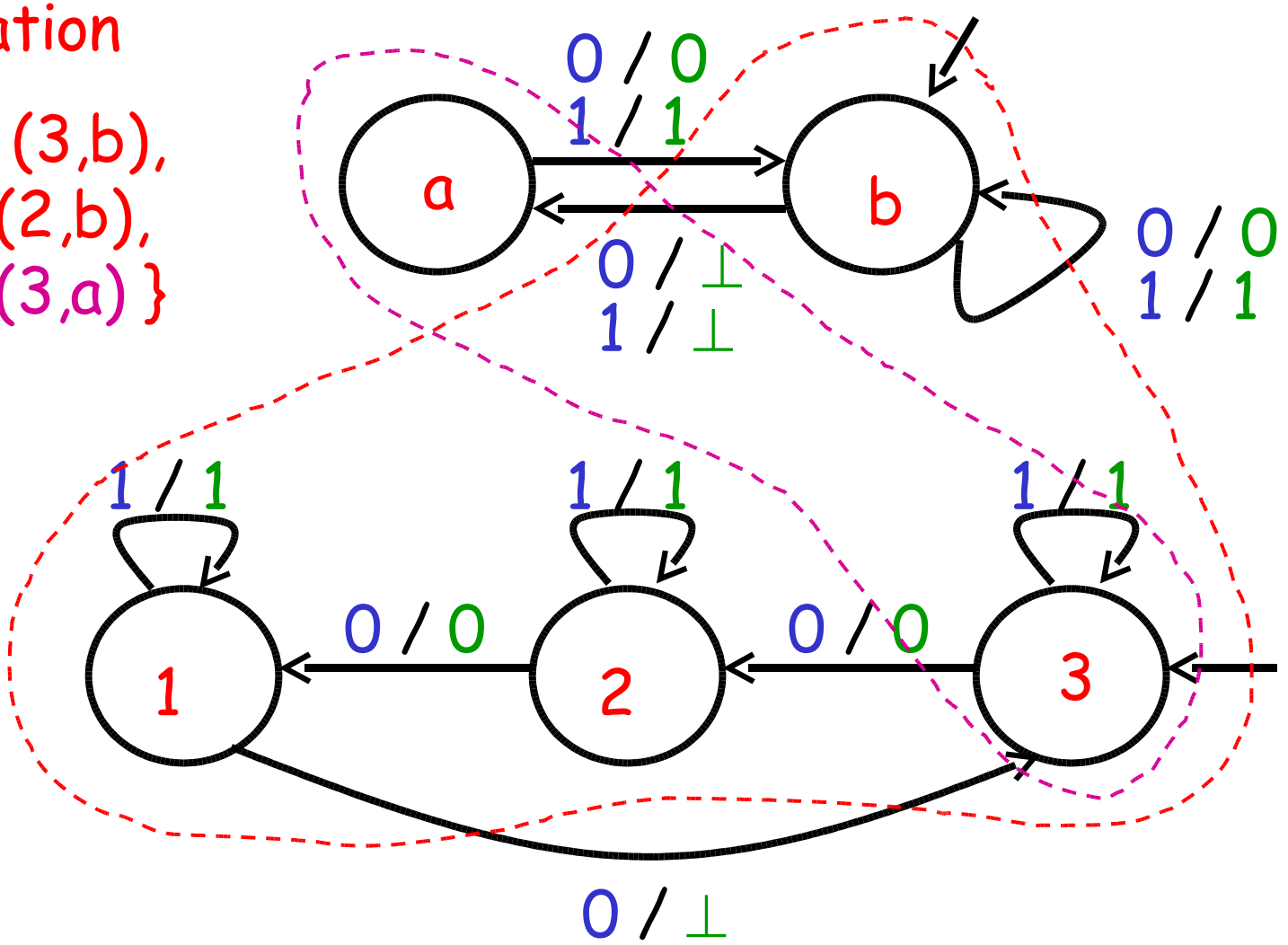


Example - (continued)

ThirdZero is simulated by NotTwice

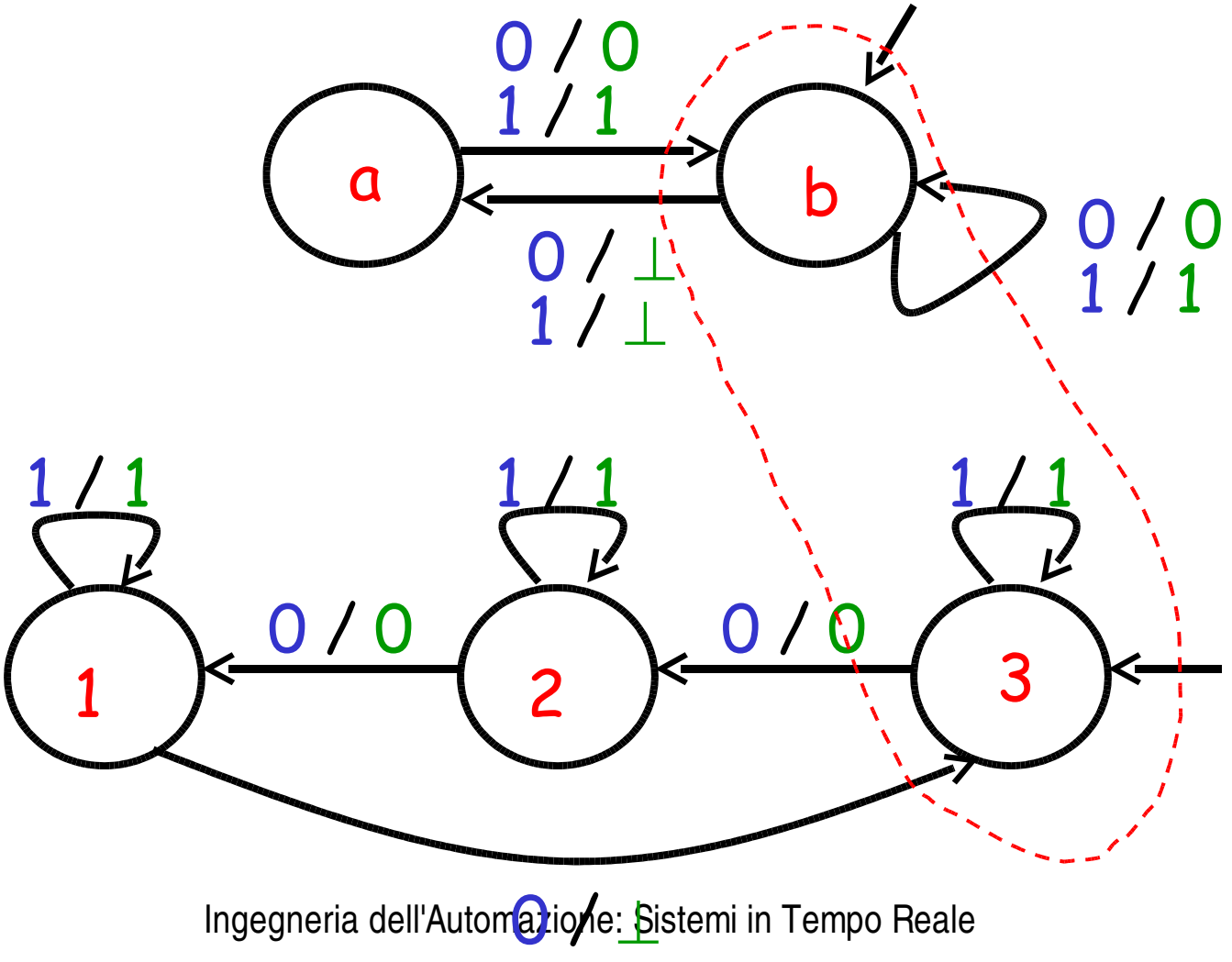
Simulation

S = { (3,b),
(2,b),
(1,b), (3,a) }



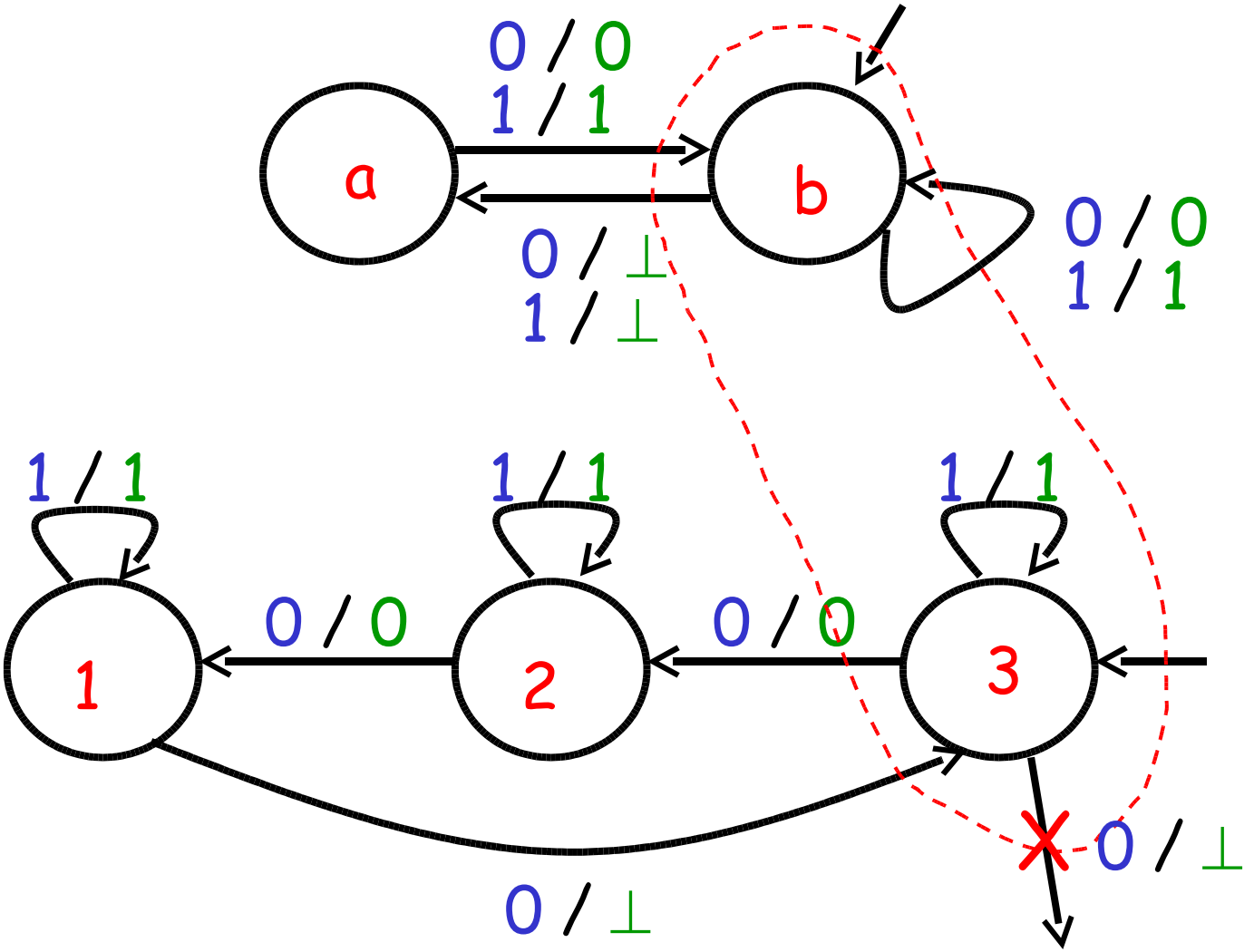
Example - (continued)

NotTwice is not simulated by ThirdZero



Example - (continued)

NotTwice is not simulated by ThirdZero



Remark

- If M2 satisfies a property, so does M1
- For instance, insofar as the property not-twice, is of interest, we can prove it on *NotTwice* and it also applies to *ThirdZero*

Remark 1

If M2 is a deterministic state machine, then

M1 is simulated by M2

iff

M1 is equivalent to M2.

Remark 2

If M2 is a nondeterministic state machine, then

M1 is simulated by M2

implies

M1 refines M2,

but M2 may not refine M1.

(Recall M1 = ThirdZero and M2 = NotTwice .)