# Ingegneria dell'Automazione - Sistemi in Tempo Reale

# *Selected topics on discrete-time and sampled-data systems*


Luigi Palopoli

palopoli@sssup.it - Tel. 050/883444

# Outline

- Systems and I/O behaviours
- Linear time-invariant Discrete-time systems and behaviours
  - Discrete Transfer-function
  - State-space representations
- Representation of a sampled-data system

# Definitions

- A timeset $\mathcal{T}$ is a subgroup of $(\mathbb{R}, +)$
  - Continuous-Time: $\mathcal{T} = \mathbb{R}$
  - Discrete-Time (DT): $\mathcal{T} = \mathbb{Z}$
  - Discrete-Event: $\mathcal{T}$ is a countable subset of $\mathbb{R}$ and there is a finite number of elements between any two elements

# Sequences

- a sequence $u$ is a mapping froma a subset of $\mathcal{T}$ to a set $\mathcal{U}$
- given an interval $I$ the set of all sequences from $I$ into $\mathcal{U}$ is denoted as:

$$\mathcal{U}^I = \{\omega \mid \omega : I \rightarrow \mathcal{U}\}$$

- Example if $\mathcal{T} = \mathbb{Z}$ the set $U^{[0,k)} = \omega(0), \ldots, \omega(k-1)$

# System

- a system (or machine) is a five tuple $\Sigma = (\mathcal{T}, \mathcal{X}, \mathcal{U}, \phi)$ consisting of:
  - a timeset $\mathcal{T}$
  - a nonempty statespace $\mathcal{X}$
  - an input-value space $\mathcal{U}$
  - a transition map $\phi : D_\phi \to \mathcal{X}$ defined on a subset $D_\phi$ of:

$$\{(\tau, \sigma, x, \omega) \mid \tau, \sigma \in \mathcal{T}, \mathbf{x} \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma, \tau)}\}$$

  - ... and some techinical axioms that we omit here.

## System with output

- A system with output is a system $\Sigma$ with:
  - a set $\mathcal{Y}$ called the measurement-value space
  - a readout map $h$:

$$h : \mathcal{T} \times \mathcal{X} \to \mathcal{Y}$$

  - it is possible to consider output maps where the ouput depends also on the input (e.g., non strictly causal systems)

$$h : \mathcal{T} \times \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$$

## Some taxonomy

- a DT system is one for which $\mathcal{T} = \mathbb{Z}$
- a system is time-invariant (TI) iff

> *for each* $\omega \in \mathcal{U}^{[\sigma,\tau)}$, *each* $x \in \mathcal{X}$, *each* $\mu \in \mathcal{T}$, *if* $\omega$ *is admissible for* $x$ *then* $\omega^\mu \in \mathcal{U}^{[\sigma+\mu,\tau+\mu)}$, $\omega^\mu = \omega(t-\mu)$ *is admissible for* $x$,
> *moreover* $\phi(\tau,\sigma,x,\omega) = \phi(t+\mu,\sigma+\mu,x,\omega^\mu)$

# FSM

- Finite State Machines (FSM)
  - $\mathcal{X}$ is finite
  - $\mathcal{U}$ is finite
  - $\mathcal{Y}$ is finite
  - $\mathcal{T}$ is a discrete-event timeset

# Trajectory

- A trajectory is a pair of functions $(\xi, \omega)$, $\xi \in \mathcal{X}^I$, $\omega \in \mathcal{U}^I$ such that

$$\xi(\tau) = \phi(\tau, \sigma, \xi(\sigma), \omega \mid_{[\sigma, \tau)})$$

  holds for each pair $\sigma, \tau \in I, \sigma < \tau$

# I/O behaviour

- an I/O behaviour is a 4-tuple $\Lambda = (\mathcal{T}, \mathcal{U}, \mathcal{Y}, \lambda)$ consisting of
  - a timeset $\mathcal{T}$
  - a nonempty control-value set $\mathcal{U}$
  - a nonempty output-value set $\mathcal{Y}$
  - a response map: $\lambda : D_\lambda \to \mathcal{Y}$ which is defined on a noneempty subset $D_\lambda$ of

$$\{(\tau, \sigma, \omega) \mid \sigma, \tau \in \mathcal{T}, \sigma \leq \tau, \omega \in \mathcal{U}^{[\sigma,\tau)}\}$$

# I/S behaviour - System

- Initialised system is a pair $(\Sigma, x_0)$, where $x_0$ is the initial state
- an Input/State (I/S) behaviour of $(\Sigma, x_0)$ is the behaviour with the same $\mathcal{T}$,$\mathcal{U}$ as $\Sigma$ and with output-value space $\mathcal{X}$, whose response map is defined on the projection

$$\{(\tau, \sigma, \omega) \mid (\tau, \sigma, x_0, \omega) \in D_\phi\}$$

$$\lambda(\tau, \sigma, \omega) = \phi(\tau, \sigma, x_0, \omega).$$

# I/O behaviour of a System

- if $(\Sigma, x_0)$ is an initialised system with output, then the I/O behaviour of $(\Sigma, x_0)$ has the same $\mathcal{T}, \mathcal{U}, \mathcal{Y}$ as $\sigma$
- the domain is the projection

$$\{(\tau, \sigma, \omega) \mid (\tau, \sigma, x_0, \omega) \in D_\phi\}$$

- and response map:

$$\lambda(\tau, \sigma, \omega) = h(\tau, \phi(\tau, \sigma, x_0, \omega))$$

## Linear systems

- A DT system $\Sigma$ is linear over the field $\mathbb{K}$ if
  - it is complete (every input is admissible for every state)
  - $\mathcal{X}$ and $\mathcal{U}$ are vector spaces
  - $\mathcal{P}(t,.,.)$ is linear for each $t \in \mathbb{Z}$ where

$$\mathcal{P}(t,x,u) = \phi(t+1,t,x,\omega)$$

- in addition, if the system has output then
  - $\mathcal{Y}$ is a vector space
  - $h(t,.)$ is linear for each $t$

## Linear DT systems

- Linearity is equivalent to the existence of two matrices $A(t)$, $B(t)$ such that

$$\mathcal{P}(t, x, u) = A(t)x + B(t)u$$

- if the system has outputs then

$$h(t, x) = C(t)x$$

  - if the output depends on $u$ then

$$h(t, x, u) = C(t)x + D(t)u$$

- if the system is TI then $A(t), B(t), C(t), D(t)$ do not depend on $t$

# Reachability

$$x(k+1) = f(x(k),\, u(k))$$

*is reachable*

iff for any $x_0, x_1$,there exists $u(k)$, $k = 0, 1, \ldots N$ such that $x(0) = x_0$ and $x(N) = x_1$

# Reachability - Linear Systems

- For $x(k+1) = Ax(k) + Bu(k)$, we have:

$$\begin{aligned} x(n) &= A^n x(0) + A^{n-1}Bu(0) + \ldots + Au(n-1) = \\ &= A^n x(0) + [BAB \ldots A^{n-1}B] \begin{bmatrix} u(n-1) \\ u(n-2) \\ \ldots u(0) \end{bmatrix} \\ &= A^n x(0) + W_c \begin{bmatrix} u(n-1) \\ u(n-2) \\ \ldots u(0) \end{bmatrix} \end{aligned}$$

- the System is reachable iff $W_c$ is invertible.

# Observability

$$x(k+1) = f(x(k),\, u(k))$$
$$y(k) = h(x(k))$$

*is observable*
iff there exists $N < \infty$ such that $x(0) = x_0$ can be determined from $y(0),\, y(1),\, \ldots,\, y(N)$.

# Observability - Linear Systems

- For $x(k+1) = Ax(k) + Bu(k), y(k) = Cx(k)$, we have:

$$y(0) = Cx(0), y(1) = CAx(0), \ldots$$

$$W_o x_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \ldots CA^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \ldots \\ y(n-1) \end{bmatrix}$$

- the System is observable iff $W_c$ is invertible.

# Linear DT I/O behaviours

- An I/O bevhaviour $\Lambda$ is linear if:
  - it is complete
  - $\mathcal{U}$, $\mathcal{Y}$ are vector spaces
  - for each $\sigma \in \mathbb{Z}$ and $\tau \in \mathbb{Z}$, with $\sigma \leq \tau$, $\lambda(\tau, \sigma, \omega)$ is linear with respect to $\omega$

# Linear DT TI I/O behaviours - I

- introduce $\delta(t)$ (kronecker delta): $\delta(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$
- a generic sequence $u(t)$ can be expressed as $u(t) = \sum_{k=\sigma}^{\tau} u(k)\delta(t-k)$
- let $h(t) = \lambda(t, 0, \delta(t))$ (pulse response)
- due to linearity and time invariance, the response to $u(t)$ can be expressed as $y(t)) = \sum_{k=\sigma}^{\tau} h(t-k)u(k) \triangleq h(k) * u(k)$ (convolution). For general sequences and noncausal systems $\sigma = -\infty, \tau = +\infty$

# The $Z-$transform

- The $Z-$transform plays for DT systems as the Laplace transform plays in CT systems
- given a sequence $e(k)$, the $Z-$ trasform $E(z)$ is given by $E(z) = \mathcal{Z}[e(t)] = \sum_{-\infty}^{+\infty} e(k) z^{-k}$ where $z$ is a complex variable

# The $Z-$transform I

- Some properties
  - Linearity: $\mathcal{Z}[\alpha f + \beta g] = \alpha F(z) + \beta G(z)$
  - Time-shift: $\mathcal{Z}[f(t-n)] = z^{-n}F(z)$
  - Convolution:
    $y(t) = h(t) * u(t) \Rightarrow Y(z) = H(z)U(z)$
  - Forward step: $\mathcal{Z}[y(t+1)] = z(Y(z) - y_0)$
  - ...please consult a basic text book...

# Transfer function

- General notations for expressing the transfer function (TF), i.e., the $Z-$transform of the pulse response
  - Polynomial representation: $H(z) = \frac{b_0+b_1 z^{-1}+\ldots b_m z^{-m}}{1+a_1 z^{-1}+\ldots+a_n z^{-n}}$

**Matlab code**:

```
num = [b0, b1, ...bm]; den
= [1, a1, ...an];
sys = tf(num,den,T);
```

  - Zero Pole representation: $H(z) = K \frac{\prod_{i=1}^{m}(z-z_i)}{\prod_{i=1}^{n}(z-p_i)}$

**Matlab code**:

```
z = [z1, ...bm]; p= [p1
...pn]; k = K;
sys = zpk(z,p,k,T);
```

# TF of a LTI DT system

- consider an initalised Single Input Single Output (SISO) system

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t),$$

with initial state $x_0 = 0$

- the pulse response is

$$y(t) = \sum_{j=0}^{t} CA^t B \rightarrow Y(z) = C(zI - A)^{-1} BU(z)$$

# Realization of a TF

- Realization is the converse problem: given a TF $H(z)$ find a system $\Sigma$ whose TF is $H(z)$
- Preliminary point: is a realization unique? NO!, Given a realization $A, B, C$ also $A', B', C'$ is a realization, where

$$A' = \begin{bmatrix} a & 0 \\ 0 & A \end{bmatrix}, B' = \begin{bmatrix} 0 \\ b \end{bmatrix}, C' = [h\, C],$$

  for any $a, h$.
- we aim at *minimal* realization (completely controllable and observable)

## Control canonical form

- Start from: $E(z) = H(z)U(z) = \frac{a(z)}{b(z)}U(z)$, where $a(z) = z^3 + a_1 z^2 + a_2 z + a_3$, $b(z) = b_0 z^3 + b_1 z^2 + b_2 z + b_3$
- Introduce $\xi = E(z)/a(z) \rightarrow U(z) = b(z)\xi$, $a(z)\xi = E(z)$
- From the properties of the $Z-$transform:

$$a(z)\xi = E(z) \Rightarrow$$

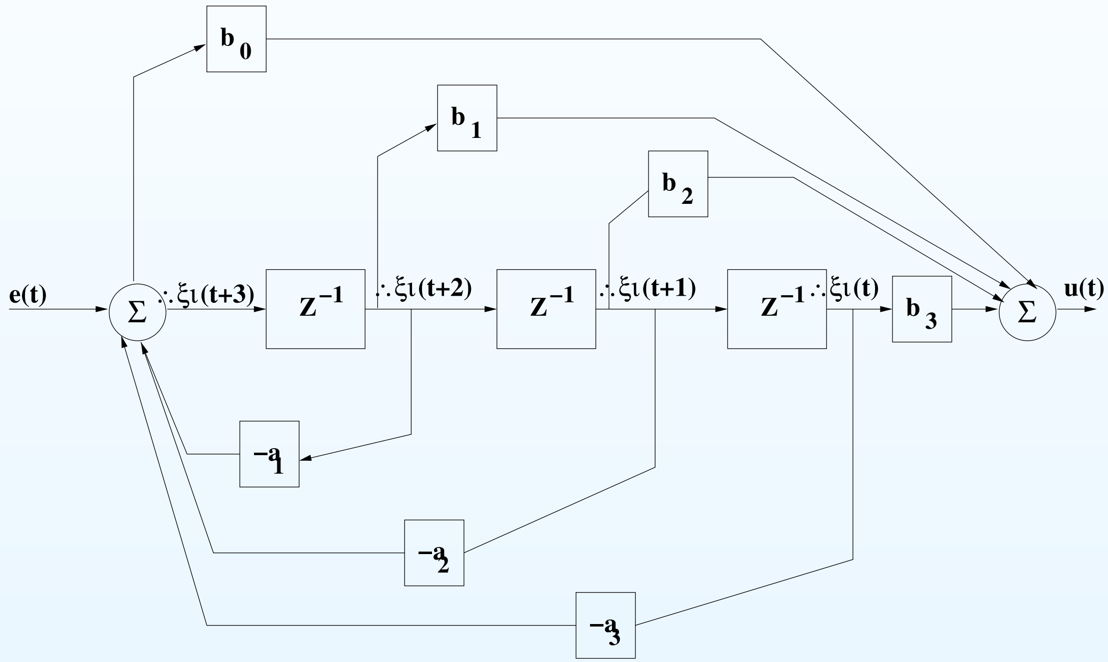
$$\xi(t+3) = e(t) - a_1\xi(t+2) - a_2\xi(t+1) - a3\xi(t)$$

$$b(z)\xi = U(z) \Rightarrow$$

$$u(t) = b_0\xi(t+3) + b_1\xi(t+2) + b2\xi(t+1) + b_3\xi(t)$$

# Control canonical form - I

- Graphical representation:



# Control canonical form - II

- Define $\xi(t+2) = x_1(t)$, $\xi(t+1) = x_2(t)$, $\xi(t) = x_3(t)$
- It is possible to write:

$$x_1(t+1) = -a_1 x_1(t) - a_2 x_2(t) - a_3 x_3(t) + e(k),$$

$$x_2(t+1) = x_1(t),\ x_3(t+1) = x_2(t)$$

$$u(t) = b_0 e(t) + (b_1 - a_1 b_0) x_1(t) +$$

$$+(b_2 - a_2 b_0) x_2(t) + (b_3 - a_3 b_0) x_3(t)$$

- The system can be written as

$$\mathbf{x}(t+1) = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e(t)$$

$$u(t) = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & b_3 - a_3 b_0 \end{bmatrix} u(t) + [b_0] u(t)$$

## Observer Canonical Form

- The difference equations can be written as:

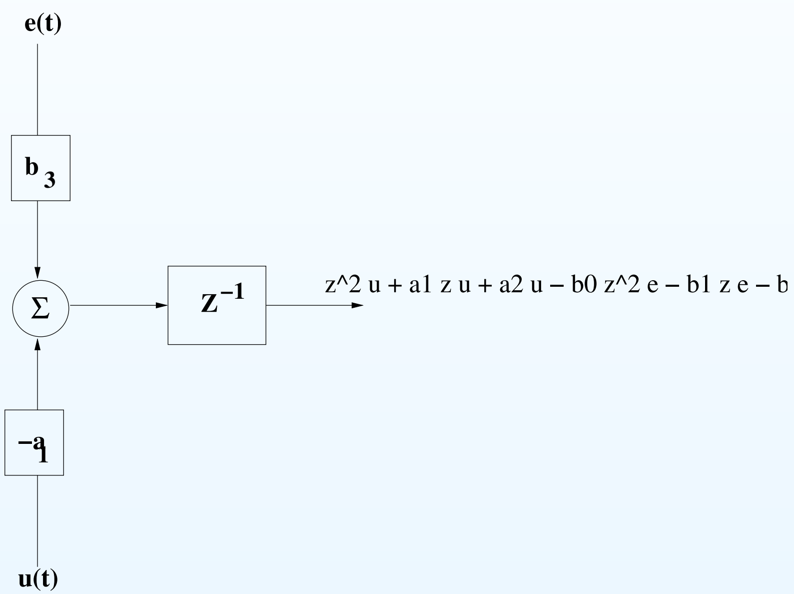
$$z^3u + a_1z^2ua_2zu + a_3u = b_0z^3e + b_1z^2e + b_2ze + b_3e$$

- We put on the lhs everything that does not depend on $z$:

$$b_3e - a_3u = z^3u + a_1z^2ua_2zu - b_0z^3e - b_1z^2e - b_2ze$$

# Observer Canonical Form - I

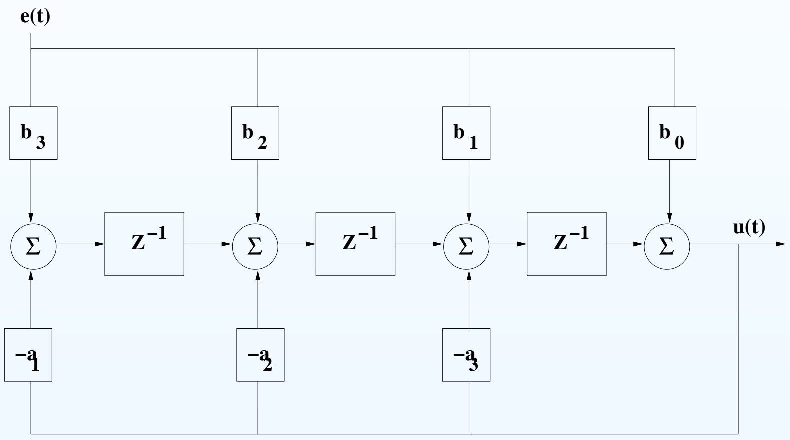
- Graphical Representation




- Hey, we can play the same game working on the output of the delay element!!

# Observer Canonical Form - II

- After some iteration we get:



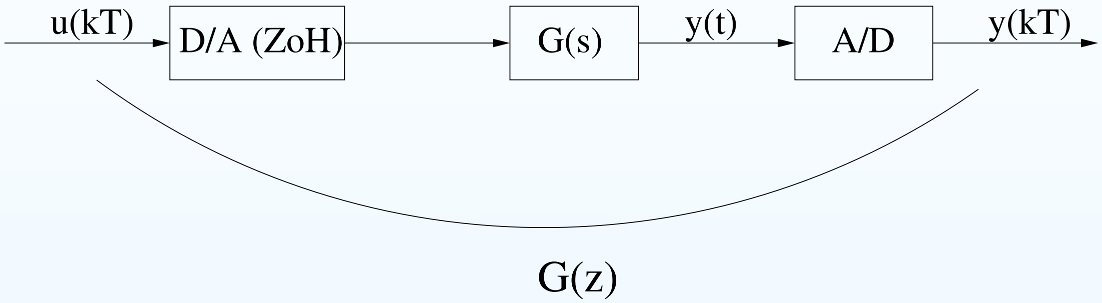

- The system can be written as $\mathbf{x}(t+1) = A_o\mathbf{x}(t) + b_o e(t), u(t) = C_o u(t) + D_o u(t)$ where:

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}, b_o = \begin{bmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ b_3 - b_0 a_3 \end{bmatrix}$$

$$C_o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D_o = [b_0]$$

# DT models of sampled-data systems

- Using $\mathcal{Z}-$transform




- Notice: $G(s)$ and $G(z)$ represent the *same* plant
- To compute $G(z)$ apply the impulse $\delta(t)$
- $U(s) = \frac{1-e^{-sT}}{s} \rightarrow Y(s) = \frac{G(s)}{s}(1-e^{-sT})$
- $G(z) = \mathcal{Z}[\mathcal{L}^{-1}[Y(s)]] = (1-z^{-1})\mathcal{Z}[\mathcal{L}^{-1}[\frac{G(s)}{s}]]$

# Example

- $G(s) = \frac{1}{s^2}$
- $G(z) = (1 - z^{-1})\mathcal{Z}[\mathcal{L}^{-1}[\frac{1}{s^3}]]$
- $\mathcal{L}^{-1}[\mathcal{L}^{-1}[\frac{1}{s^3}] = \frac{t^2}{2}$
- $\mathcal{Z}[\frac{(kT)^2}{2}] = \frac{T^2}{2}\mathcal{Z}[k^2]$
- Property of $\mathcal{Z}-$transform: $\mathcal{Z}[k^{h+1}] = -z\frac{d\mathcal{Z}[k^h]}{dz}$
- Using the above: $\mathcal{Z}[\frac{T^2 z(z+1)}{2(z-1)^3}] \Rightarrow G(z) = \mathcal{Z}[\frac{T^2(z+1)}{2(z-1)^2}]$