Ingegneria dell 'Automazione - Sistemi in Tempo Reale

Selected topics on discrete-time and sampled-data systems

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Outline

- Multirate and periodic systems
- LMI methods
- Introduction to sampled data systems

Semidefinite programming

- A semidefinite program is an optimisation problem having the following form

$$
\begin{aligned}
\min \quad & c^T x \\
\text{subj. to} \quad & F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n \preceq 0 \\
& Ax = b
\end{aligned}
$$

- $F(x) = F_0 + x_1 F_1 + \ldots + x_n F_n \preceq 0$, $F_i = F_i^T$ is called a Linear Matrix Inequality (LMI)

Semidefinite Programming - I

- Semidefinite programs belong to the class of convex programs
- If a point is a local optimum then it is also a global optimum
- Efficient (polynomial time) solution methods exist
- There are different tools to express and solve optimisation problems
  - Sedumi is one of the best open source options
  - Other tools (such as yalmip) are used as front-end

Example I

- Analyse Lyapunov Stability of $\dot{x} = Ax$

$$\begin{array}{ll} \min & 0 \\ \text{subj. to} & A^T P + PA \preceq 0 \\ & P \succeq 0 \end{array}$$

- ....

Example I - continued

```
A = [-1 2;0 -2];
yalmip('clear');

P = sdpvar(2,2,'symmetric');
F = lmi('P>0') + lmi('A''*P+P''*A<0');

solution = solvesdp(F);

Pfeasible = double(P);
eig(Pfeasible)
eig(A'*Pfeasible+Pfeasible'*A)
```

Example II

- Analyse Lyapunov Stability of $\dot{x} = (A + \Delta)x$ for $\Delta = diag(\delta_1, \delta_2)$, $\delta_1 \in [\underline{\delta_1},\, \overline{\delta_2}]$, $\delta_2 \in [\underline{\delta_2},\, \overline{\delta_2}]$
- Necessary and sufficient condition: Existence of a lyapunov function for all Δ
- Sufficient condition: the same Lyapunov function for all Δ. It can be shown to be equivalent to the following semidefinite program

$$
\begin{array}{ll}
\min & 0 \\
\text{subj. to} & (A+\Delta_1)^T P + P(A+\Delta_1) \preceq 0 \\
 & (A+\Delta_2)^T P + P(A+\Delta_2) \preceq 0 \\
 & (A+\Delta_3)^T P + P(A+\Delta_3) \preceq 0 \\
 & (A+\Delta_4)^T P + P(A+\Delta_4) \preceq 0 \\
 & P \succeq 0
\end{array}
$$

- $\Delta_1 = diag(\underline{\delta}_1, \underline{\delta}_2)$, $\Delta_2 = diag(\overline{\delta}_1, \underline{\delta}_2)$, $\Delta_3 = diag(\underline{\delta}_1, \overline{\delta}_2)$, $\Delta_4 = diag(\overline{\delta}_1, \overline{\delta}_2)$

Example II - continued

```
A = [-1 2;0 -2];d1_min = -0.5; d2_min = -0.5;
d1_max = 0.5; d2_max = 0.5;
D1 = diag(d1_min, d2_min); D2 = diag(d1_max, d2_min);
D3 = diag(d1_min, d2_max); D4 = diag(d1_max, d2_max);
yalmip('clear');
P = sdpvar(2,2,'symmetric');
F = lmi('P>0') + lmi('(A+D1)''*P+P''*(A+D1)<0');
F = F + lmi('(A+D2)''*P+P''*(A+D2)<0');
F = F + lmi('(A+D3)''*P+P''*(A+D3)<0');
F = F + lmi('(A+D4)''*P+P''*(A+D4)<0');
solution = solvesdp(F);eig(double(P));
eig(A'*double(P)+double(P)'*A)
```

Example III

- Choose a gain γ s.t. the discrete time $x^+ = (A + b\gamma)x$ be stable
- We can use the following program:

$$\begin{array}{ll} \min & 0 \\ \text{subj. to} & (A+b\gamma)^T P (A+b\gamma) - P \preceq 0 \\ & P \succ 0 \end{array}$$

- This is not yet a semidefinite program. However we can use the following result (Schur complements):
  $(A + b\gamma)^T P (A + b\gamma) - P \preceq 0, P \succ 0$ is equivalent to:
  $$\begin{bmatrix} -W & AW + b\gamma W \\ AW + b\gamma W & -W \end{bmatrix} \preceq 0, \text{ where } W = P^{-1}$$

Example III - continued

- Introduce $Y = \gamma W$
- Now we can just solve:

$$\begin{array}{ll} \min & 0 \\ \text{subj. to} & \begin{bmatrix} -W & (AW + bY)^T \\ AW + bY & -W \end{bmatrix} \preceq 0 \end{array}$$

- The gain can be found by: $\gamma = YW^{-1}$

Example III - continued

```
A = [-1 2;0 -2]; b=[1;-2];
yalmip('clear');
eps=0.1;
W = sdpvar(2,2,'symmetric');
Y = sdpvar(1,2);
F = lmi('W >eps*eye(2,2)')+lmi('[-W (A*W+b*Y)''; A*W+b*Y -W]<-ep
solution = solvesdp(F);
```

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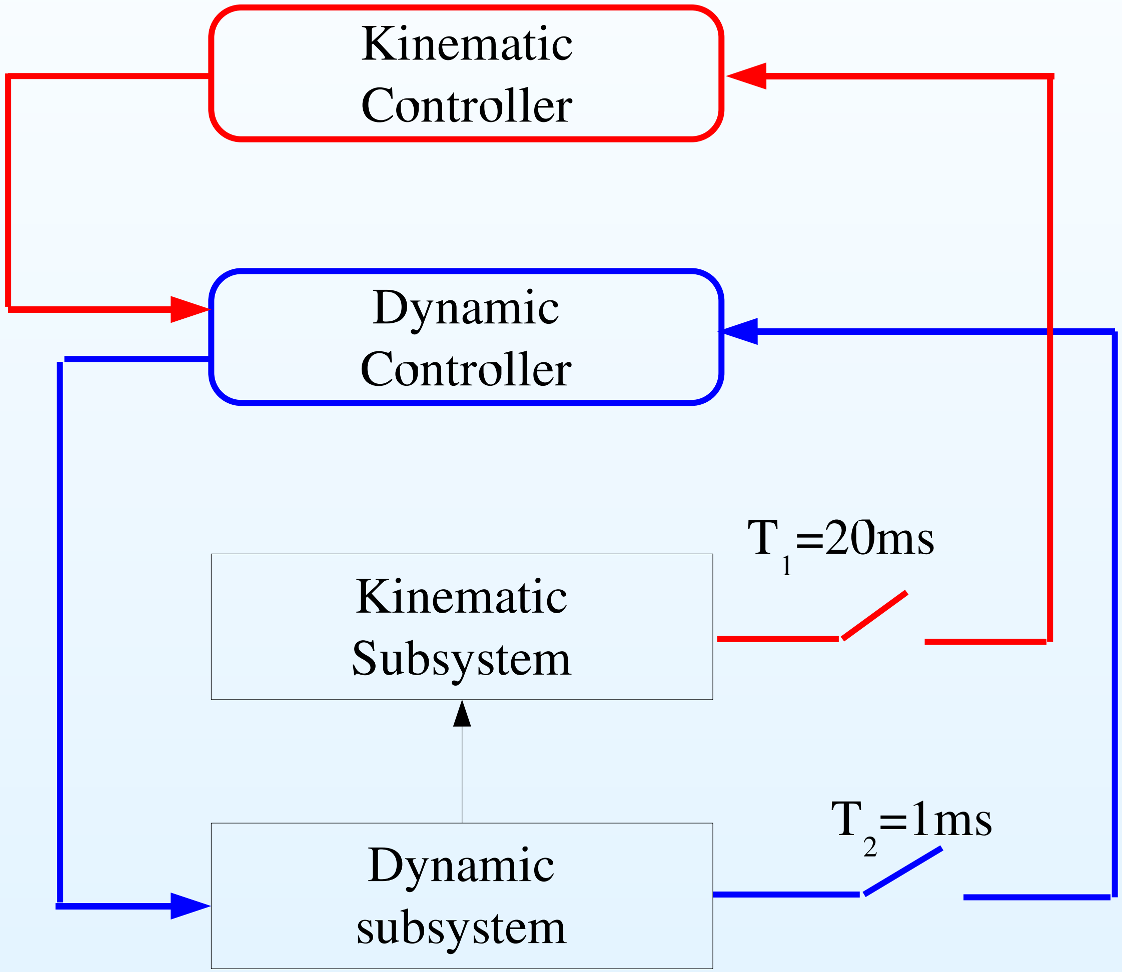
Outline

- Multirate and periodic systems
  - Modeling
  - Stability
  - LTI representations

Motivation

- Frequently a system is comprised of several subsystems
- In some cases data can be collected with different frequencies
- It can be useful to design several interconnected controllers activated at different frequencies

Example


Kinematic
Controller
Dynamic
Controller
T
1
=20ms
Kinematic
Subsystem
T
2
=1ms
Dynamic
subsystem

Example (continued)

- Assumption: sampling periods are integre multiples of the smallest period
- Let $x_1 \in \mathbb{R}^{n_1}$ be kinematic variables, and $x_2(k) \in \mathbb{R}^{n_2}$ be dynamic variables
- Equation (sampled at the slowest period):

$$x_1(k+1) = A_{1,\,1}\, x_1(k) + A_{1,\,2}\, x_2(k)$$
$$x_2(k+1) = A_{2,\,2}\, x_2(k) + B_2\, u(k)$$

- The different periods for data measurements can be modeled assuming a periodic C matrix:
  $y(k) = C(k)\left[\begin{smallmatrix} x_1(k) \\ x_2(k) \end{smallmatrix}\right]$
  where
  $$C(k) = \begin{cases} I_{(n_1+n_2)\times(n_1+n_2)} & k = hT, T = T_1/T_2, h \in \mathbb{N} \\ [\,0\ I_{n_1\times n_1}\,], & \text{otherwise.} \end{cases}$$
- Observe: the dimensions of the output space change in time...

Example (continued)

- Closed loop dynamics
  - we can assume that data communicated from the kinematic to dynamic controller are held throughout the period T_2
  - to modify this sample and hold intracontroller mechanism we need a further state variable
  - The resulting clodes loop dyanmics will be something like:

$$x(k+1) = A_c(k)x(k)$$

where matrix A_c is periodic with period $T = T_1/T_2$

Periodic systems

- From the example above it is clear that multirate control systems can easily be modeled by the use of periodic (i.e. periodically timevarying systems)
- Plants can be modeled as:

  $$x(k+1) = A(k)x_(k) + B(k)u(k)$$
  $$y(k) = C(k)x(k) + D(k)u(k)$$
  where $A(k+T) = A(k), B(k+T) = B(k), C(k+T) = C(k), D(k+T) = D(k)$

- In particular closed loop autonomous systems are modeled as $x(k+1) = A(k)x(k), A(k+T) = A(k)$

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- Multirate and periodic systems
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Stability results

- Consider the autonomous periodic system

$$x(k+1) = A(k)x(k),\ A(k+T) = A(k) \tag{1}$$

- Consider the subsequence $\tilde{x}(h) = x(k_0 + hT)$ for some k_0; the sequence is described by

$$\tilde{x}(h+1) = \tilde{A}_{k_0}\tilde{x}(h),\ \text{where } \tilde{A}_{k_0} = A(k_0 + T - 1)A(k_0 + T - 2)...A(k_0) \tag{2}$$

- Fact: System (1) is [Asymptotically] stable $\Leftrightarrow$ System (2) is
  - $\Rightarrow$ obvious
  - $\Leftarrow$ (intuition) The intersampling evolution (due to the finitiness of the period) is norm-bounded: there exist a constant M such that $x(k) \leq M\tilde{x}(h), \forall k \in [hT, h(T+T) - 1[$, so if $\tilde{x}(h)$ vanishes so dows $x(k)$

Stability results

- More formally...:
  - Lemma: the eigenvalues of the matrix $\tilde{A}_{k_0} = A(k_0 + T - 1)A(k_0 + T - 2)...A(k_0)$, called *monodromy matrix*, are indpendent of k_0
  - System (1) is stable if and only if the monodromy matrix is Schur-stable (i.e. its eignecvalues are all in the unit circle)
- Based on the observation that the stability can evaluated on the periodic subsampled sequence we can build lyapunov inspired criteria...
- Example: the system is A.S. if there exist periodic definite matrices $P(k)$ that respect the following linear matrix inequalities:
$$\begin{array}{l} A^T(0)P(1)A(0) - P(0) \prec 0 \\ A^T(1)P(2)A(1) - P(1) \prec 0 \\ \dots \\ A^T(T-1)P(0)A(T-1) - P(T-1) \prec 0 \end{array}$$
- With some trick (Schur complements) the above can be used as a synthesis tool for finding stabising gains (by using Linear Matrix inequalities solvers)

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LTI representations

- Consider the system:

$$
\begin{aligned}
&x(k+1) = A(k)x_(k) + B(k)u(k) \\
&y(k) = C(k)x(k) + D(k)u(k) \\
&\text{where } A(k+T) = A(k), B(k+T) = B(k), C(k+T) = C(k), D(k+T) = D(k) \\
&x \in \mathbb{R}^n, y \in \mathbb{R}^p u \in \mathbb{R}^m
\end{aligned}
\tag{3}
$$

- We want to find LTI equivalent systems for which important properties are preserved

Lifting

- The idea of considering periodically extracted subsequences for stability can be exploited also in general terms
- The lifted representation for the system is achieved by the following steps:
  - sampling the evolution of the state with the same periodicitiy of the system
  - grouping inputs used throughout a period and output emitted throughout a period into augemented vectors
  - Lifted state: $\hat{x}^{(h_0)}(h) = x(h_0 + hT)$
  - Lifted inputs:

$$\hat{u}^{(h_0)}(k) = \begin{bmatrix} u(h_0 + hT) \\ u(h_0 + hT + 1) \\ \dots \\ u(h_0 + hT + T - 1) \end{bmatrix}, \ \hat{y}^{(h_0)}(k) = \begin{bmatrix} y(h_0 + hT) \\ y(h_0 + hT + 1) \\ \dots \\ y(h_0 + hT + T - 1) \end{bmatrix}$$

Lifting (continued)

- the lifted system is described by (h_0 specification omitted):

$$\begin{aligned}\hat{x}(h+1) &= \hat{A}\hat{x}(h) + \hat{b}\hat{u}(h) \\ \hat{y}(h) &= \hat{C}\hat{x}(h) + \hat{D}\hat{u}(h)\end{aligned} \tag{4}$$

where

$\hat{A} = A(h_0 + T - 1) \dots A(h_0)$

$\hat{b} = [\hat{b}_1 \hat{b}_2 \dots \hat{b}_T]$, with $\hat{b}_i = A(h_0 + T - 1) \dots A(h_0 + i)B(h_0 + i - 1$

$\hat{C} = \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \dots \\ \hat{c}_T \end{bmatrix}$, with $\hat{c}_i = C(h_0 + i - 1)A(h_0 + i - 2) \dots A(h_0)$

$\hat{D} = \{\hat{d}_{i,j}\}$ with $i, j = 1, 2, \dots T$ and

$$\hat{d}_{i,j} = \begin{cases} C(h_0 + i - 1)A(h_0 + i - 2 \dots A(h_0 + j)B(h_0 + j - 1) & i < j \\ 0, & i \geq j \end{cases}$$

Lifting (continued)

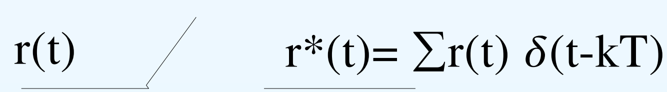
- The reachability space of $[A(), B()]$ a $h = h_0$ coincides with the reachibility space of $[\hat{A}, \hat{B}]$
- The pair $[A(), B()]$ is stabilisable iff the pair $[\hat{A}, \hat{B}]$ is
- The observability space of $[A(), B()]$ at $h = h_0$ coincides with the observability subspace of $[\hat{A}, \hat{C}]$
- The pair $[A(), C()]$ is detectable iff the pair $[\hat{A}, \hat{B}]$ is

Sampled-data systems

- So far we have seen discrete-time systems
- Computer controlled systems are actually a mix of discrete-time and continuous time systems
- We need to understand the interaction between different components

Ideal sampler

- Ideal sampling can intuitively be seen as generated by multiplying a signal by a sequence of dirac's δ



Properties of δ

- $\int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = f(a)$
- $\int_{-\infty}^{t} \delta(\tau)d\tau = 1 \rightarrow \mathcal{L}[\delta(t)] = 1$

$\mathcal{L}$-trasform of r^*

- Using the above properties it is possible to write:

$$\mathcal{L}[r^*(t)] = \int_{-\infty}^{+\infty} r^*(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} r(\tau)\delta(\tau - kT)d\tau =$$
$$= \sum_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(\tau)\delta(\tau - kT)d\tau = \sum_{-\infty}^{+\infty} r(kT)e^{-sKT} = R^*(s)$$

- The $\mathcal{L}$-transform of the sampled-data signal r can be achieved from the $\mathcal{Z}$-transform of the sequence $r(kT)$ by choosing $z = e^{-sT}$
- Backward or forward shifting yields different samples: the sampling operation is not time-invariant (but it is linear)

Spectrum of the Sampled Signal

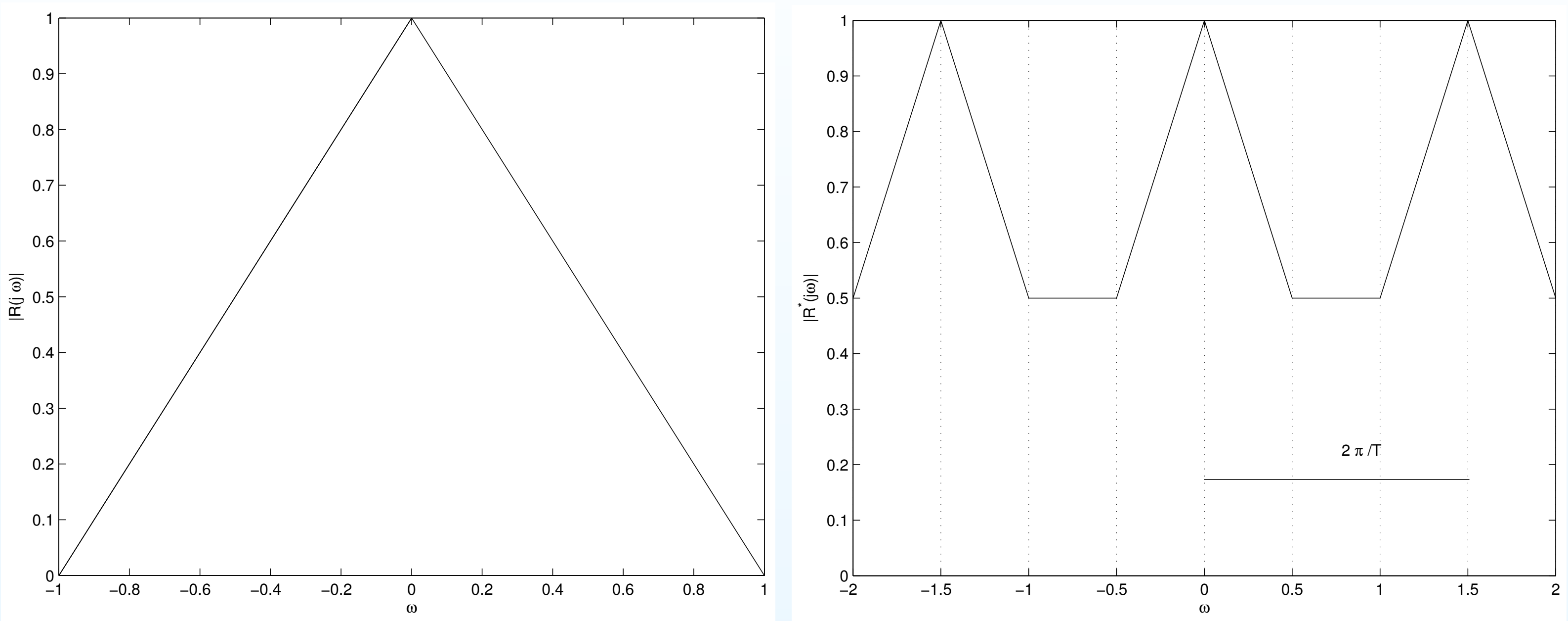
- Fourier Series transform of the sampling signal

$$\sum_{-\infty}^{+\infty} \delta(t-kT) = \sum_{-\infty}^{+\infty} C_n e^{j2n\pi t/T}$$
$$C_n = \int_{-T/2}^{T/2} \sum_{-\infty}^{+\infty} \delta(t-kT) e^{-j2n\pi t/T} d\tau = \int_{-T/2}^{T/2} \delta(t) e^{-j2n\pi t/T} d\tau = \frac{1}{T} \rightarrow$$
$$\sum_{-\infty}^{+\infty} \delta(t-kT) = \frac{1}{T} \sum_{-\infty}^{+\infty} e^{j2n\pi t/T}$$

- Spectrum of r^*

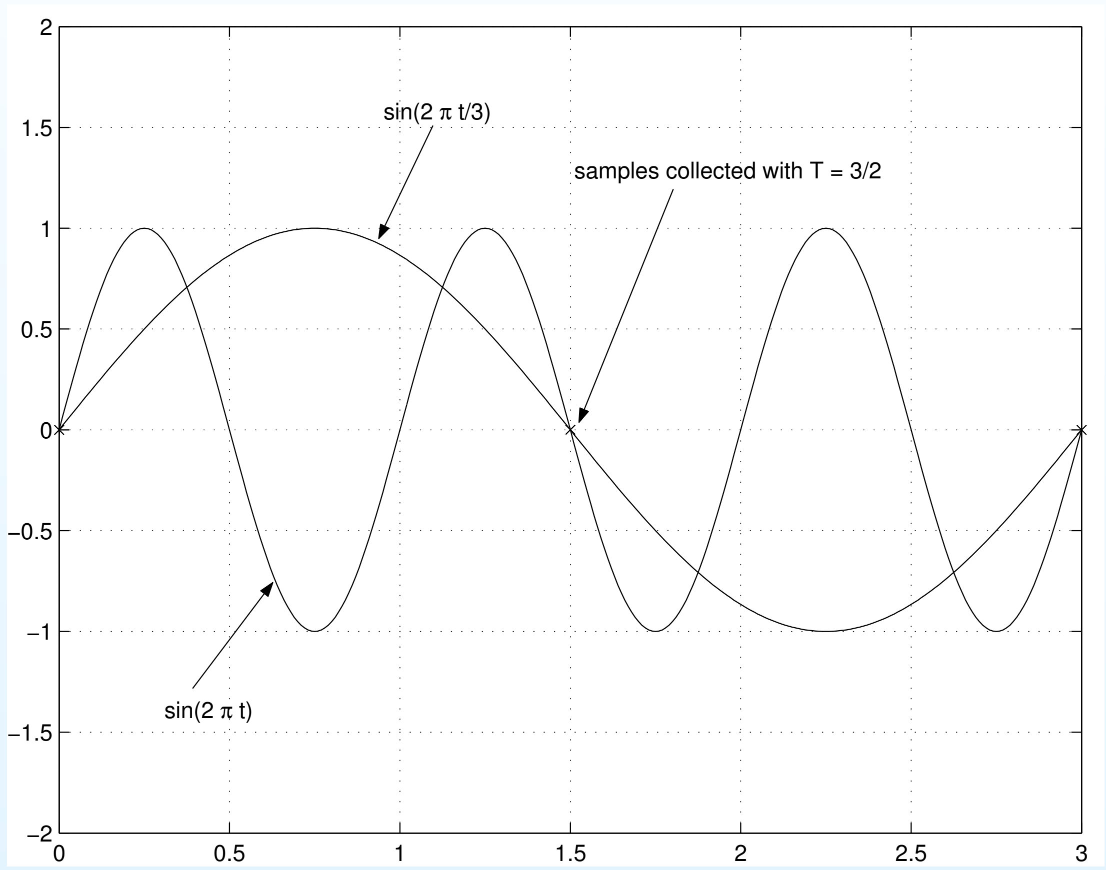
$$\mathcal{L}[r^*(t)] = \int_{-\infty}^{+\infty} r(t) \frac{1}{T} \sum_{-\infty}^{+\infty} e^{j2n\pi t/T} e^{-sT} dt =$$
$$= \frac{1}{T} \sum_{-\infty}^{+\infty} r(t) \int_{-\infty}^{+\infty} e^{-(s-j2n\pi/T)t} dt =$$
$$= \frac{1}{T} \sum_{-\infty}^{+\infty} R(s - j2\pi n/T)$$

Example



Aliasing

The spectrum might be altered (i.e., signal not attainable from samples!)



Shannon theorem

- *If the signal has a finite badwidth then the signal can be reconstructed from samples collected with a period such that* $\frac{1}{T} \geq 2B$
- Band-limited signals have infinite duration; many signals of interest have infinite bandwidth
- Typically a low-pass filter is used to de-emphasize higher frequencies

Data Extrapolation

- If the following hypotheses hold
  - the signal has *limited* bandwidth B
  - the signal is sampled at frequency $f_s = \frac{1}{T} \geq 2B$
- then the signal can be reconstructed using an ideal lowpass filter $L(s)$ with

$$|L(j\omega)| = \begin{cases} T & \text{if } \omega \in [-\frac{\pi}{T}, \frac{\pi}{T}] \\ 0 & \text{elsewhere.} \end{cases}$$

- Signal $l(t)$ is given by:

$$l(t) = \frac{1}{2\pi} \int_{-pi/T}^{pi/T} T e^{j\omega T} d\omega = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}(\pi t/T)$$

Data Extrapolation I

- The reconstructed signal is computed as follows:

$$r(t) = r^*(t) * l(t) = \int_{-\infty}^{+\infty} r(\tau) \sum \delta(\tau - kT) \text{sinc} \frac{\pi(t-\tau)}{T} d\tau =$$
$$= \sum_{-\infty}^{+\infty} r(kT) \text{sinc} \frac{\pi(t-kT)}{T}$$

- The function sinc is not causal and has infinite duration
- In communication applications
  - The duration problem can be solved truncating the signal
  - The causality problem can be solved introducing a delay and collecting some sample before the reconstruction
- Not *viable* in control applications since large delays jeopardise stability

Extrapolation *via* ZOH

- Zoh transfer function

$$Zoh(j\omega) = \frac{1-e^{j\omega T}}{j\omega} = e^{-j\omega T/2}\left\{\frac{e^{j\omega T/2} - ^{-j\omega T/2}}{2j}\right\}\frac{2j}{j\omega} =$$
$$= Te^{j\omega T/2}\text{sinc}(\frac{\omega T}{2})$$

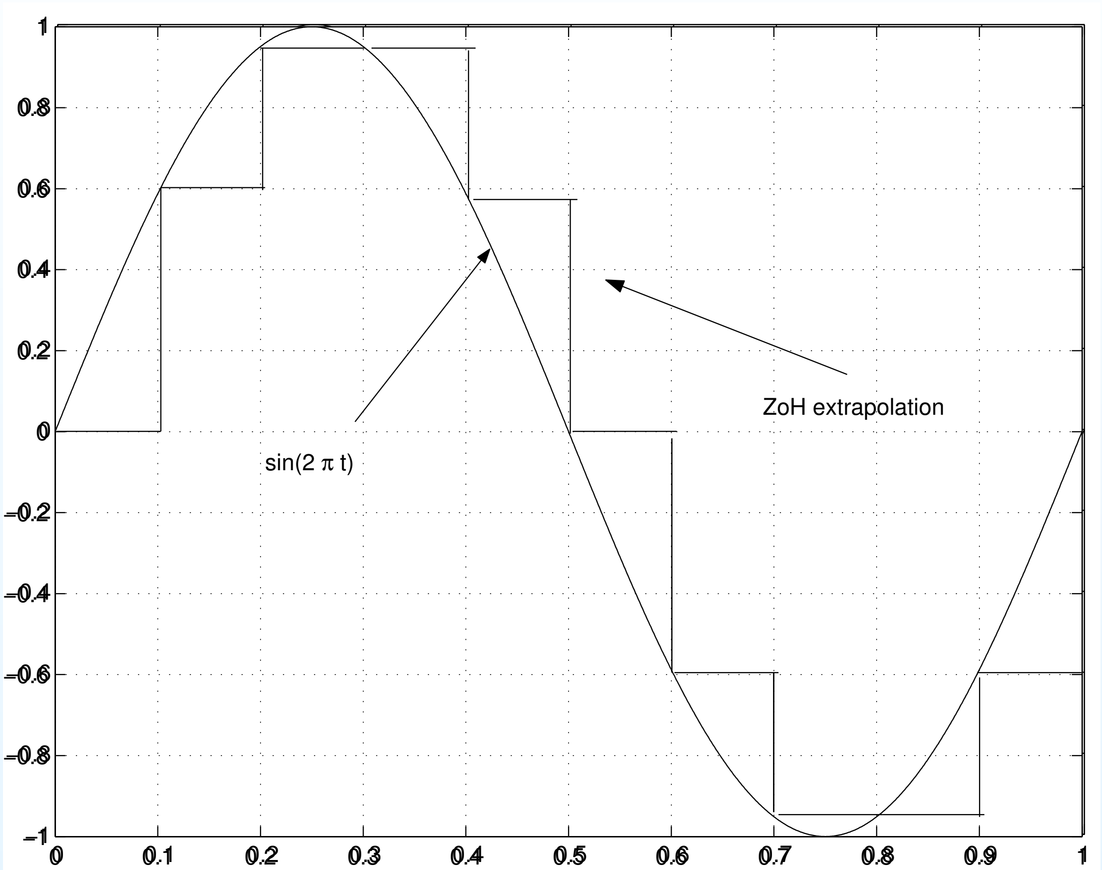
- Amplitude:

$$|Zoh(j\omega)| = T|\text{sinc}(\frac{\omega T}{2})|$$

- Phase:

$$\angle Zoh(j\omega) = -\frac{\omega T}{2}$$

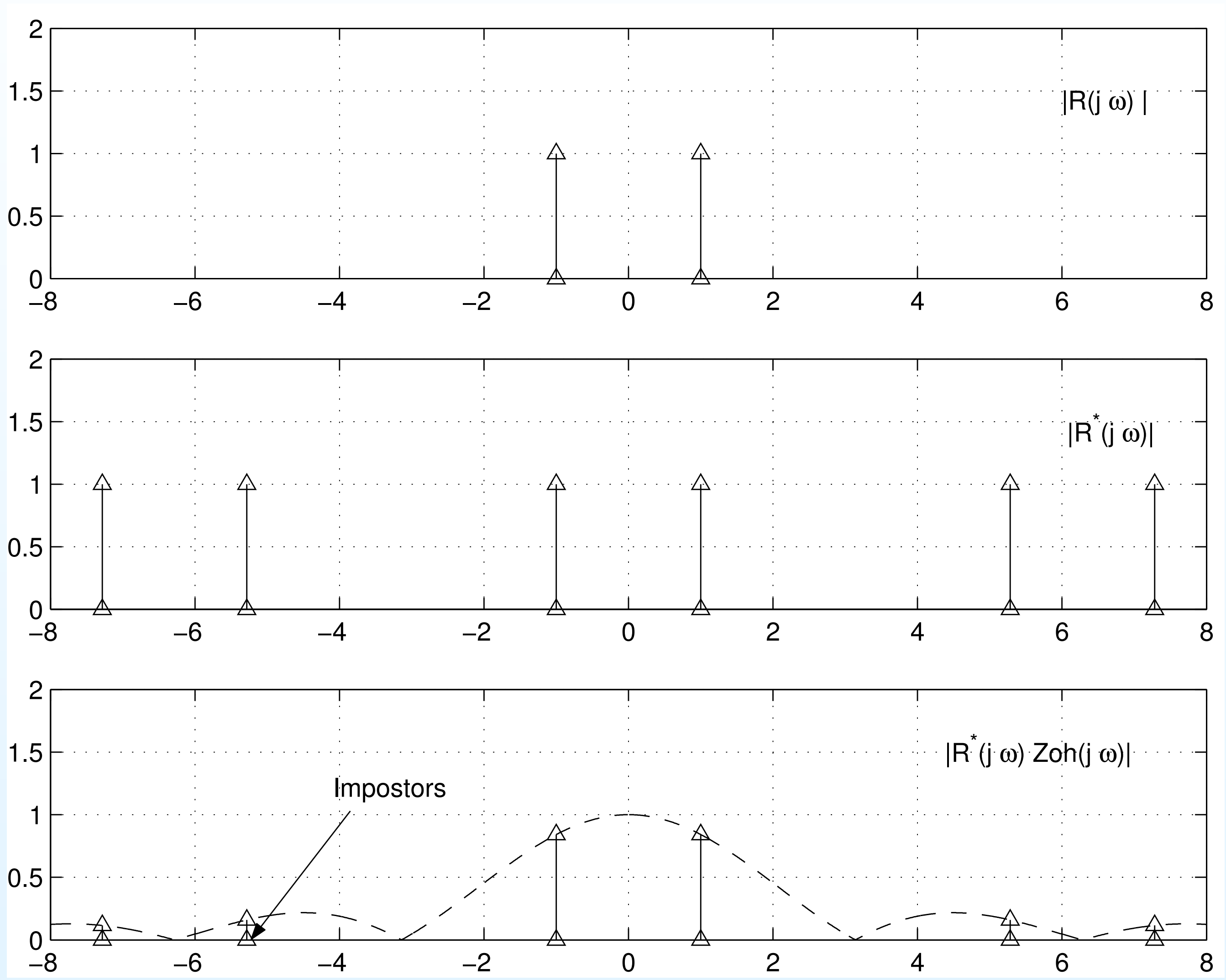
Sinusoidal signal



Example

- Consider the signal $r(t) = \frac{1}{\pi}\sin(t)$
- The spectrum is given by $R(j\omega) = j(\delta(\omega - 1) - \delta(\omega + 1))$
- Suppose we sample it with period $T = 1$
- The spectrum of $r^*(t) = 1/T \sum_{k=-\infty}^{+\infty} R(j\omega - 2n\pi)$

Example - I



Example - II

- Sampling and ZoH extrapolation orignate spurious harmonics (called impostors)
- Taking adavantage of the low pass behaviour of the plant we can restrict to the first harmonic

$$v_1(t) = \frac{1}{\pi}\mathsf{sinc}(T/2)\sin(t - T/2)$$

- The sample-and-hold operation can be thought of (at a first approximation) as the introduction of delay of $T/2$