Ingegneria dell ’Automazione - Sistemi in Tempo Reale

Selected topics on discrete-time and sampled-data systems

Luigi Palopoli

palopoli@sssup.it - Tel. 050/883444

Outline

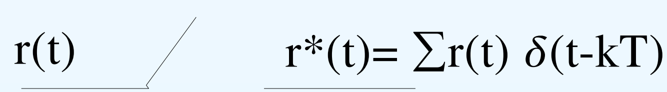
- Introduction to sampled data systems

Sampled-data systems

- So far we have seen discrete-time systems
- Computer controlled systems are actually a mix of discrete-time and continuous time systems
- We need to understand the interaction between different components

Ideal sampler

- Ideal sampling can intuitively be seen as generated by multiplying a signal by a sequence of dirac's δ



Properties of δ

- $\int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = f(a)$
- $\int_{-\infty}^{t} \delta(\tau)d\tau = 1 \rightarrow \mathcal{L}[\delta(t)] = 1$

$\mathcal{L}$-trasform of r^*

- Using the above properties it is possible to write:
  $\mathcal{L}[r^*(t)] = \int_{-\infty}^{+\infty} r^*(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} r(\tau)\delta(\tau - kT)d\tau =$
  $= \sum_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r(\tau)\delta(\tau - kT)d\tau = \sum_{-\infty}^{+\infty} r(kT)e^{-sKT} = R^*(s)$
- The $\mathcal{L}$-transform of the sampled-data signal r can be achieved from the $\mathcal{Z}$-transform of the sequence $r(kT)$ by choosing $z = e^{-sT}$
- Backward or forward shifting yields different samples: the sampling operation is not time-invariant (but it is linear)

Spectrum of the Sampled Signal

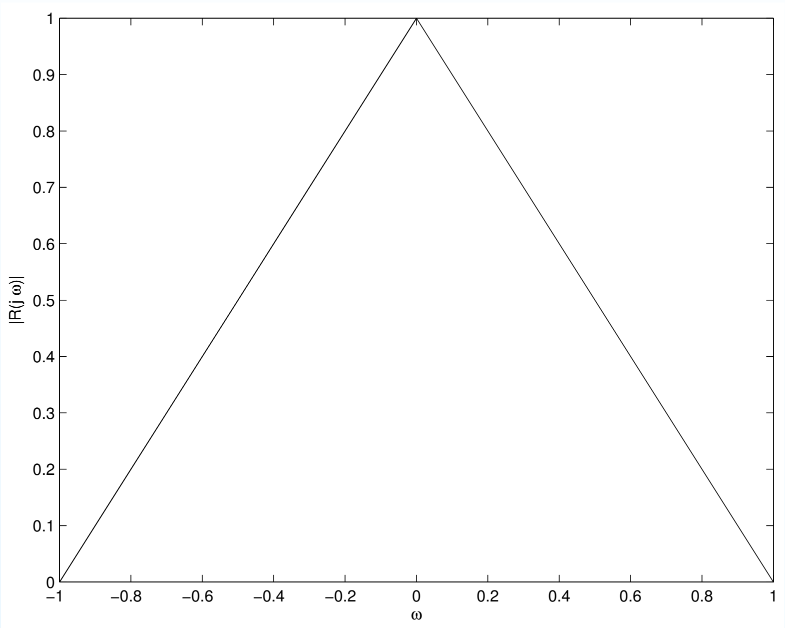
- Fourier Series transform of the sampling signal

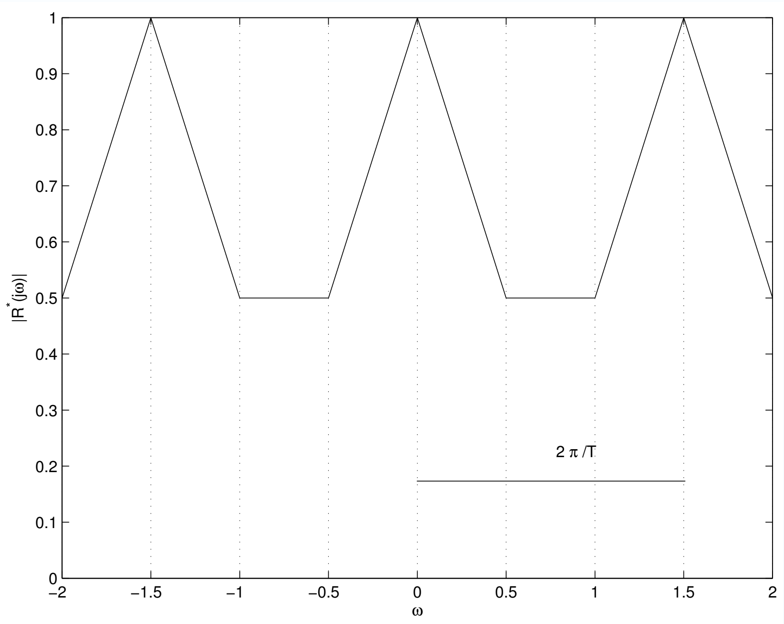
$$\sum_{-\infty}^{+\infty} \delta(t - kT) = \sum_{-\infty}^{+\infty} C_n e^{j2n\pi t/T}$$
$$C_n = \int_{-T/2}^{T/2} \sum_{-\infty}^{+\infty} \delta(t - kT) e^{-j2n\pi t/T} d\tau = \int_{-T/2}^{T/2} \delta(t) e^{-j2n\pi t/T} d\tau = \frac{1}{T} \rightarrow$$
$$\sum_{-\infty}^{+\infty} \delta(t - kT) = \frac{1}{T} \sum_{-\infty}^{+\infty} e^{j2n\pi t/T}$$

- Spectrum of r^*

$$\mathcal{L}[r^*(t)] = \int_{-\infty}^{+\infty} r(t) \frac{1}{T} \sum_{-\infty}^{+\infty} e^{j2n\pi t/T} e^{-sT} dt =$$
$$= \frac{1}{T} \sum_{-\infty}^{+\infty} r(t) \int_{-\infty}^{+\infty} e^{-(s - j2n\pi/T)t} dt =$$
$$= \frac{1}{T} \sum_{-\infty}^{+\infty} R(s - j2\pi n/T)$$

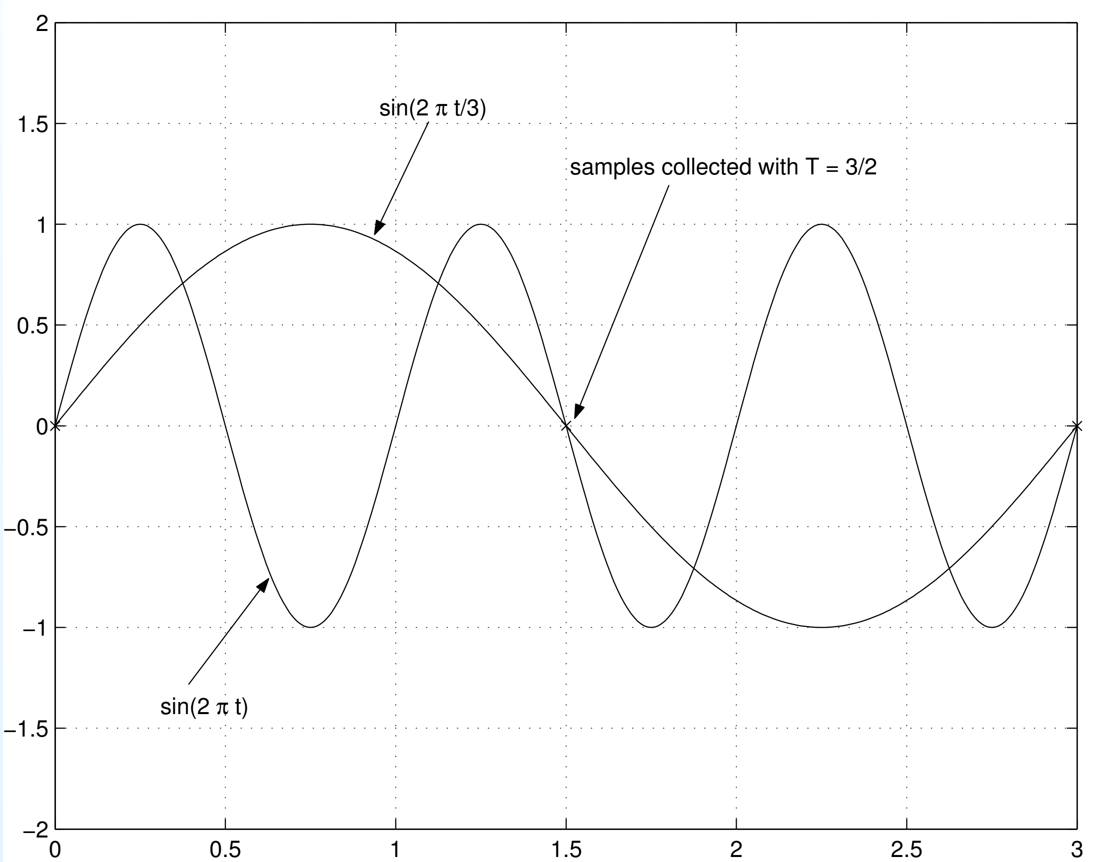
Example





Aliasing

The spectrum might be altered (i.e., signal not attainable from samples!)



Shannon theorem

- *If the signal has a finite badwidth then the signal can be reconstructed from samples collected with a period such that* $\frac{1}{T} \geq 2B$
- Band-limited signals have infinite duration; many signals of interest have infinite bandwidth
- Typically a low-pass filter is used to de-emphasize higher frequencies

Data Extrapolation

- If the following hypotheses hold
  - the signal has *limited* bandwidth B
  - the signal is sampled at frequency $f_s = \frac{1}{T} \geq 2B$
- then the signal can be reconstructed using an ideal lowpass filter $L(s)$ with

$$|L(j\omega)| = \begin{cases} T & \text{if } \omega \in [-\frac{\pi}{T}, \frac{\pi}{T}] \\ 0 & \text{elsewhere.} \end{cases}$$

- Signal $l(t)$ is given by:

$$l(t) = \frac{1}{2\pi} \int_{-pi/T}^{pi/T} T e^{j\omega T} d\omega = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}(\pi t/T)$$

Data Extrapolation I

- The reconstructed signal is computed as follows:

$$r(t) = r^*(t) * l(t) = \int_{-\infty}^{+\infty} r(\tau) \sum \delta(\tau - kT) \text{sinc} \frac{\pi(t-\tau)}{T} d\tau =$$
$$= \sum_{-\infty}^{+\infty} r(kT) \text{sinc} \frac{\pi(t-kT)}{T}$$

- The function sinc is not causal and has infinite duration
- In communication applications
  - The duration problem can be solved truncating the signal
  - The causality problem can be solved introducing a delay and collecting some sample before the reconstruction
- Not *viable* in control applications since large delays jeopardise stability

Extrapolation *via* ZOH

- Zoh transfer function

$$Zoh(j\omega) = \frac{1-e^{j\omega T}}{j\omega} = e^{-j\omega T/2}\left\{\frac{e^{j\omega T/2} - {}^{-j\omega T/2}}{2j}\right\}\frac{2j}{j\omega} =$$
$$= Te^{j\omega T/2}\text{sinc}(\frac{\omega T}{2})$$

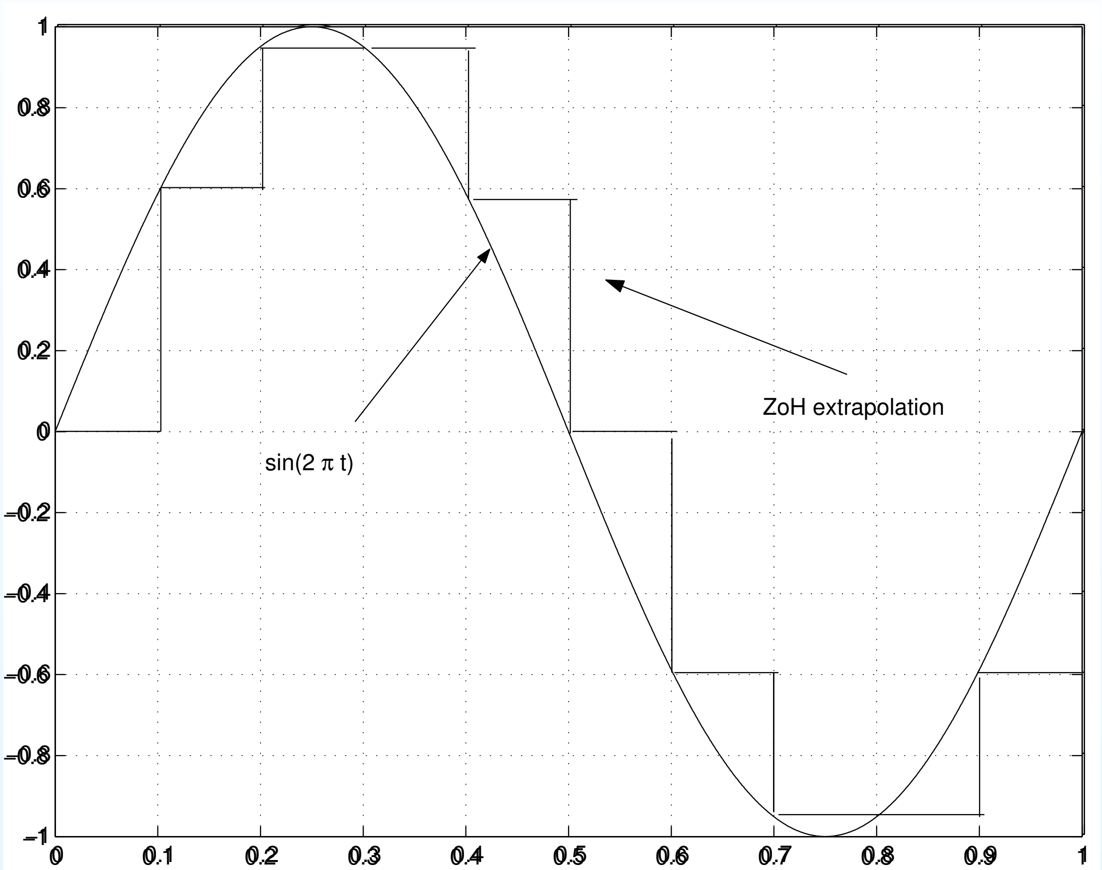
- Amplitude:

$$|Zoh(j\omega)| = T|\text{sinc}(\frac{\omega T}{2})|$$

- Phase:

$$\angle Zoh(j\omega) = -\frac{\omega T}{2}$$

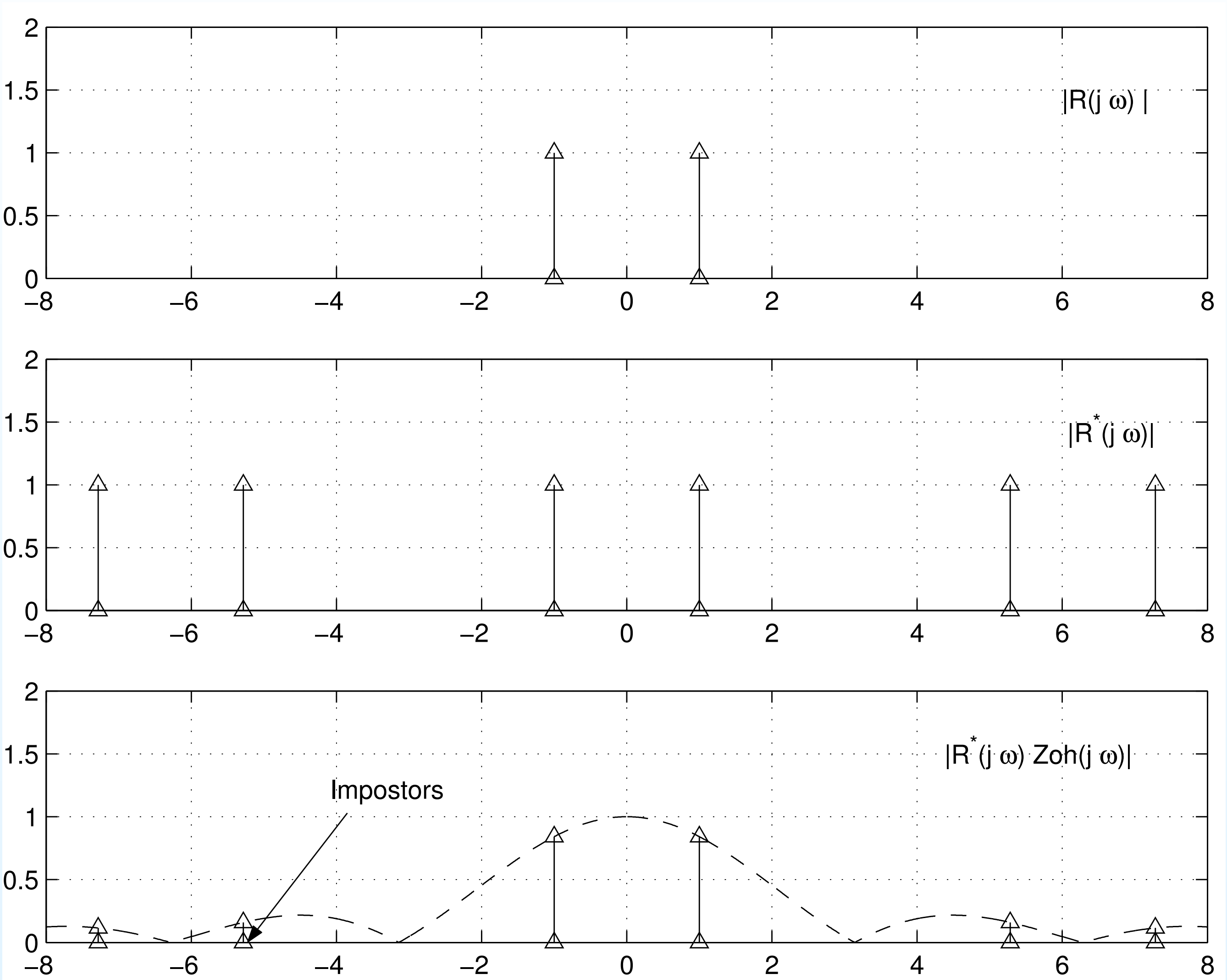
Sinusoidal signal



Example

- Consider the signal $r(t) = \frac{1}{\pi}\sin(t)$
- The spectrum is given by $R(j\omega) = j(\delta(\omega - 1) - \delta(\omega + 1))$
- Suppose we sample it with period $T = 1$
- The spectrum of $r^*(t) = 1/T \sum_{k=-\infty}^{+\infty} R(j\omega - 2n\pi)$

Example - I



Example - II

- Sampling and ZoH extrapolation orignate spurious harmonics (called impostors)
- Taking adavantage of the low pass behaviour of the plant we can restrict to the first harmonic

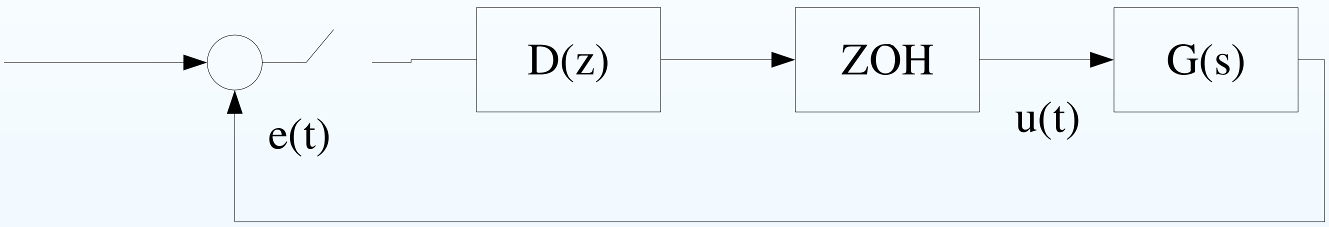
$$v_1(t) = \frac{1}{\pi}\text{sinc}(T/2)\sin(t - T/2)$$

- The sample-and-hold operation can be thought of (at a first approximation) as the introduction of delay of $T/2$

Outline

- Block diagram analysis of sampled-data systems

Stucture of a digital control system



Heterogeneous diagram: the different blocks have different meaning

Homegeneous blocks

- We aim at a block diagram where blocks are homogeneous
- Consider a sampled signal $e^*(t)$ (a sequence of δ)
- We can model the block $D(z)$ as if it operated on periodic spectrum signal (i.e., transforming sequences of δ into sequences of δ) whereas $D(z)$ operates on sequences of numbers
- The transfer function is given by:

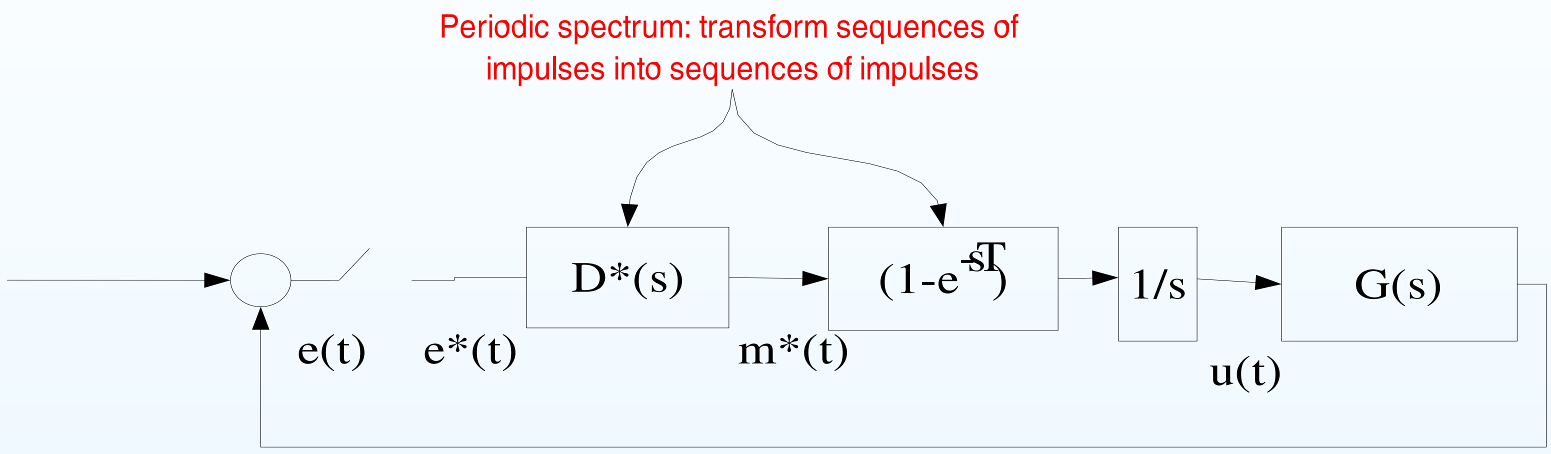
$$D^*(s) = \int_0^{+\infty} \sum_k \delta(t - kT) d(kT) e^{-sT} dt =$$
$$= \sum_k \int_0^{+\infty} \delta(t - kT) d(kT) e^{-sT} dt = \sum_k d(kT) e^{-(sT)k} = D(z)|_{z=e^{sT}}$$

- A useful fact

$$\{E^*(s)G(s)\}^* = E^*(s)G^*(s)$$

(the periodicity of $E^*(s)$ is crucial in the proof).

A different block diagram



In this case , blocks are homegeneuos

A strange transfer function

- From the diagram we read:

$$E(s) = R(s) - Y(s)$$
$$M^*(s) = D^*(s)E^*(s)$$
$$U(s) = M^*(s)\left[\frac{1-e^{-sT}}{s}\right]$$
$$Y(s) = G(s)U(s),$$

- ...from which

$$E^*(s) = R^*(s) - Y^*(s)$$
$$U^*(s) = M^*(s)\text{samples} \rightarrow \text{ZoH} \rightarrow \text{sampling} \rightarrow \text{samples}$$
$$Y^*(s) = \{G(s)U(s)\}^* = \left\{G(s)M^*(s)\left[\frac{1-e^{-sT}}{s}\right]\right\}^* =$$
$$= \left\{\frac{G(s)}{s}M^*(s)(1-e^{-sT})\right\}^*, \text{Using}\{E^*(s)G(s)\}^* = E^*(s)G^*(s)) \text{ we get}$$
$$Y^*(s) = (1-e^{-sT})M^*(s)\left\{\frac{G(s)}{s}\right\}^*$$

A strange transfer function - I

- using $M^*(s) = D^*(s)E^*(s)$, we get

$$Y^*(s) = (1 - e^{-sT})D^*(s)E^*(s)\left\{\frac{G(s)}{s}\right\}^* =$$
$$(1 - e^{-sT}D^*(s)\left\{\frac{G(s)}{s}\right\}^*(R^*(s) - Y^*(s))$$

- setting $H^*(s) = (1 - e^{-sT})D^*(s)\left\{\frac{G(s)}{s}\right\}^*$, we get

$$Y^*(s) = \frac{H^*(s)}{H^*(s) + 1}R^*(s)$$

- This is the usual form for a transfer function!!!
- Note, though, that the sampling operation is not time-invariant: this is a lucky coincidence!

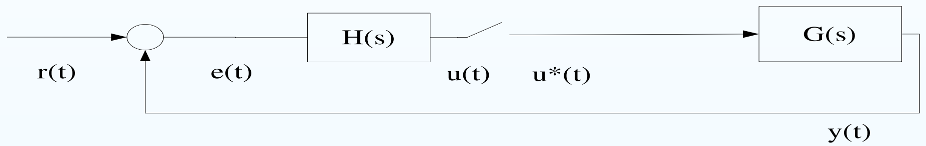
A numeric example

- Plant: $G(s) = \frac{a}{s+a}$
- Controller: $u(kT) = u((k-1)T) + k_0 e(kT) \rightarrow D(z) = \frac{k_0 z}{z-1}$
- $D^*(s) = D(z)|_{z=e^{sT}} = \frac{k_0 e^{sT}}{e^{sT}-1}$
- $H^*(s) = (1-e^{-sT})D^*(s)\left\{\frac{G(s)}{s}\right\}^* = (1-e^{-sT})D^*(s)\left\{\frac{1}{s} - \frac{1}{s+a}\right\}^* =$
  $(1-e^{-sT})D^*(s)\left(\frac{1}{1-e^{-sT}} - \frac{1}{1-e^{-aT}e^{-sT}}\right)$
- Now we are able to compute staic and dyanmic responses on H^*
- It is easy to show that

$$Y(s) = \frac{(1-e^{-sT})R^*(s)D^*(s)}{1+H^*(s)}\frac{G(s)}{s}$$

- From the above it is possible to compute the intersample behaviour. Notice that the first factor produces a train of δ impulses: in the intersampling the systems generates a sum of scaled and traslated step responses.

A different case

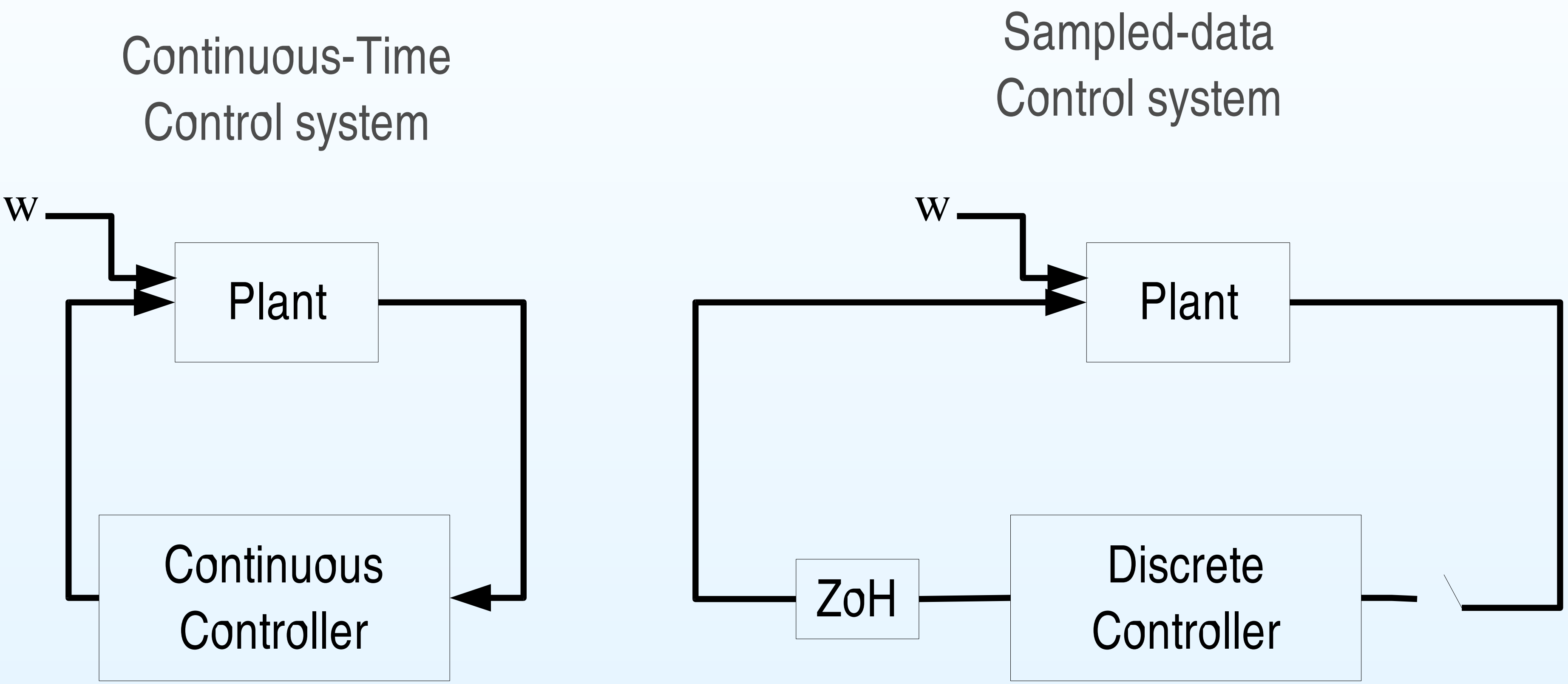


- In a case like this we get: $Y^*(s) = \frac{\{H(s)R(s)\}^*}{1+\{H(s)G(s)\}^*} G^*(s)$
- we haven't got any transfer function structure (R does not enter the system only *via* its samples.

Outline

- Sample-rate selection
  - Shannon theorem
  - smoothness of responses
  - Noise rejection
  - Robustness

Block-diagrams



Continuous-time system

- Let the closed loop equation of the system be:

$$\dot{x} = Fx + G_1\dot{w}$$

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  - $w(0) = 0$
  - $w(t) - w(s)$ is a Gaussian process with mean 0 and variance $(t-s)R_w$
  - for all times $0 < t_1 < t_2 < \dots t_n$, $w(t_1)$, $w(t_2) - w(t_1)$, $\dots$, $w(t_n) - w(t_{n-1})$ are independent

Continuous-time system -I

- Property: $\dot{w}$ is gaussian with mean 0 and variance R_w
- Assume that increments dw are incorrelated with x and with w

CT system (Mean)

- If the initial state has nonzero mean then
  $$\frac{dm}{dt} = Fm$$
  $$m(0) = m_0$$
  where $m(t)$ is the mean of $x(t)$.

CT system (continued)

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- Expressing variations of xx^T (from now on use x for $\tilde{x}$):

$$d\,xx^T = xx^T - (x+dx)(x+dx)^T = xdx^T + dxx^T + dxdx^T$$

CT system (continued)

- ...from which

$$dxx^T = x(Fx)^T dt + x(G_1 dw)^T dt + Fxx^T + G_1 dwx^T + (Fx)(Fx)^T dt^2 + (Fx)(G_1 dw)^T$$
$$+(G_1 dw)(Fx)^T dt^2 + (G_1 dw)(G_1 dw)^T$$

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- Taking expected values of both sides and considering that 1) w is incorrelated with x, 2) $E(dwdw^T) = R_w dt$ we get

$$\frac{dP}{dt} = PF^T + FP + G_1 R_w G_1^T$$

CT system (continued)

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- Taking the limit $dt \to 0$:

$$\dot{P} = PF^T + FP + G_1 R_w G_1^T$$

Covariance

- Definition
  $r(s,t) = Cov\{x(t), x(s)\} = E\{(x(t) - m(t))^T (x(s) - m(s))\}$
- Let $s \geq t$, the system evolution is given by:
  $x(s) = e^{A(s-t)}x(t) + \int_s^t e^{A(s-\tau)}w(\tau)d\tau$
- Computing the expectation value and because $x(t)$ is incorralated from $w(t)$ we get

$$r(s,t) = e^{A(s-t)}P(t), s \geq t$$

Example

- First order systems: $\frac{dx}{dt} = fx + g_1 w, f < 0$
  $var\{x(t_0)\} = r_0, mean(x(t_0)) = m(0) = m_0$
- Mean value:
$$\frac{dm}{dt} = fm \rightarrow m(t) = m_0 e^{f(t-t_0)}$$
- Covariance function
  - Differential equation, let $r_1 = g_1^2 R_w$,
$$\dot{P} = 2fP + g_1^2 R_w, P(t_0) = r_0 \rightarrow P(t) = e^{2f(t-t_0)} r_0 + \frac{r_1}{2f}(e^{2f(t-t_0)} - 1)$$
  - Assuming $f < 0$ and $m_0 = 0$, we get: $r(s,t) = \frac{r_1}{2f} e^{f|t-s|}$
- the process is asymptotically stationary: $r(\tau) = \frac{r_1}{2f} e^{f|\tau|}$
- the spectral density is $\phi(\omega) = \frac{r_1}{2\pi} \frac{1}{\omega^2 + a^2}$

Sampled-data systems

- Sampled-data equation:

$$x((k+1)T) = \Phi(T)x(k) + e(kT)$$
$$\Phi(t) = \int_0^T e^{A\tau)} d\tau$$
$$e(kT) = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} G_1 w(\tau) d\tau$$

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$$m(k+1) = \Phi(T)m(k), m(0) = m_0$$

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- Mean value Evolution:

$$m(k+1) = \Phi(T)m(k), m(0) = m_0$$

- Covariance evolution:

$$r_{xx}(k,h) = cov(x(k), x(h)) = E\{\tilde{x}(k)\tilde{x}(h)^T\}, \text{ where } \tilde{x} = x - m$$
$$\tilde{x}((k+1)T) = \Phi(T)\tilde{x}(kT) + e(kT)$$

Sampled-data systems

- Introduce $P(k) = cov(x(k), x(k))$ (T implied):

$$\tilde{x}(k+1)\tilde{x}^T(k+1) = \Phi\tilde{x}(k)\tilde{x}(k)^T\Phi^T + \Phi\tilde{x}(k)e(k)^T +$$
$$+e(k)(\Phi\tilde{x}(k))^T + e(k)e(k)^T$$

Taking the expectation and considering that e(k) and x(k) are independent...

$$P(k+1) = \Phi P(k)\Phi^T + E\{e(k)e(k)^T\}$$

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- Considering that $e(kT) = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}G_1 w(\tau)d\tau$, we get:

$$C_d = E(e(kT)e(kT)^T) = \int_0^T \Phi(\tau)G_1 R_w G_1^T \Phi(\tau)d\tau$$

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- To compute the covariance:

  $\tilde{x}(k+1)\tilde{x}(k)^T = (\Phi\tilde{x}k + e(k))\tilde{x}(k)....$

  Because e(k) has zero mean and is independent from x(k)...

  $r_{xx}(k+1,k) = \Phi P(k)$ from which

  $r_{xx}(h,k) = \Phi^{h-k}P(k), h \geq k$

Example

- First order systems: $\frac{dx}{dt} = fx + u + g_1 w, f > 0$
  $var\{x(t_0)\} = r_0, mean(x(t_0)) = m(0) = m_0$
- Design a discrete-time controller such that the equivalent CT system has a pole at $s = s_0$
- First order systems (ZoH for outputs):
  $x(k+1) = e^{fT}x(k) + \frac{e^{fT}-1}{f}u(k) + \int_{kT}^{(k+1)T} g_1 w(\tau)d\tau$
  $var\{0)\} = r_0, mean(x(0)) = m(0) = m_0$
- Choose $u(k) = kx(k)$ such that $e^{fT} + \frac{(e^{fT}-1)}{f} = e^{s_0 T}$
- Mean value:

$$m(k+1) = e^{s_0 T}m(k), m(0) = m_0 \rightarrow m(k) = e^{s_0(k-k_0)T}m_0$$

Example (continued)

- Covariance function
  - Computation of $P(k)$

$$P(k+1) = e^{2s_0T}P(k) + C_d$$
$$C_d = \int_0^T e^{2f\tau}G_1^2R_d d\tau = \frac{G_1^2R_w}{2f}(e^{2fT}-1) = \frac{r_1}{2f}(e^{2fT}-1)$$
$$P(k) = e^{2s_0(k-k_0)T}r_0 + C_d\frac{1-e^{2s_0T(k-k_0)}}{1-e^{2s_0T}}$$

  - Steady state ($k \to \infty$):

$$m(k) \to 0$$
$$P(k) \to \frac{C_d}{1-e^{2s_0T}}$$
$$r_{xx}(k,h) = \frac{C_d e^{s_0(k-h)T}}{1-e^{2s_0T}}$$

- Compare $r_{xx}(0)$ with what the value $r_{xx}^c(0)$ that we found for the continuous-time case:

$$\frac{r_{xx}(0)}{r_{xx}^c(0)} = \frac{(e^{2fT}-1)}{1-e^{2s_0T}}$$

, which is increasing with period.

Outline

- Sample-rate selection
  - Shannon theorem
  - smoothness of responses
  - Noise rejection
  - Robustness

Robustness

- Consider a first order system: $\dot{x} = ax + bu$
- We consider a robustness problems
  - b is known with uncertainity $b = \tilde{b} + db$
- Sample with period T and design so that the closed loop poles are at $e^{s_0 T}$, $s_0 < 0$
- DT system:

$$x((k+1)T) = e^{aT} x(kT) + \frac{e^{aT} - 1}{a} bu(kT)$$

- Feedback: $u(kT) = \gamma x(kT)$ s.t.

$$e^{aT} + \gamma \frac{e^{aT} - 1}{a} b = e^{s_0 T}$$

Robustness with respect to db

- Stability $|e^{aT} + \gamma\frac{e^{aT}-1}{a}(b+db)| \leq 1$
- From which:

$$1 - e^{s_0 T} \geq \gamma\frac{e^{aT}-1}{a}db \geq -1 - e^{s_0 T}$$
$$\gamma\frac{e^{aT}-1}{a} = \frac{e^{s_0 T}-e^{aT}}{b}$$
$$\frac{1-e^{s_0 T}}{e^{s_0 T}-e^{aT}} \geq \frac{db}{b} \geq -\frac{1+e^{s_0 T}}{e^{s_0 T}-e^{aT}}$$

- The measure of the maximum relative deviation:

$$\mu_{db} = \left|\frac{2}{e^{s_0 T} - e^{aT}}\right|$$