Ingegneria dell 'Automazione - Sistemi in Tempo Reale

Selected topics on discrete-time and sampled-data systems

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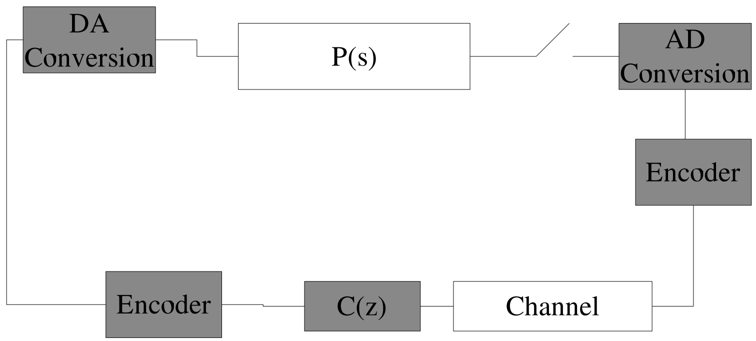
Outline

- Effects of quantisation
- Parameters round-off

Quantisation in control

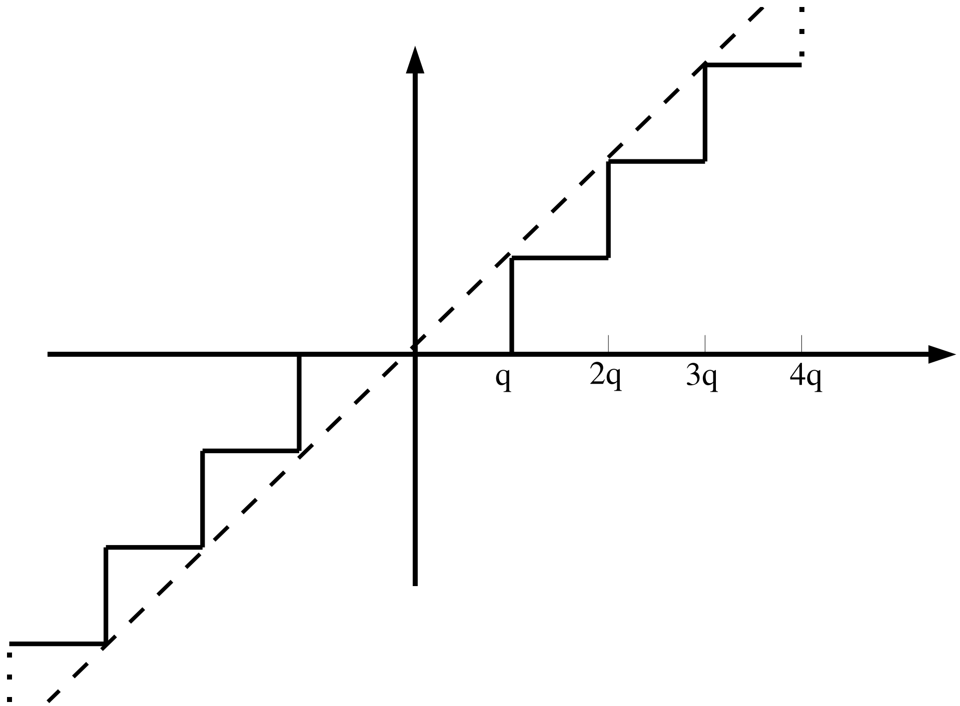


- Quantisation is used
  - communications through digital channels
  - DA/AD conversion
  - finite precision representation of numbers

What does a quantiser do?

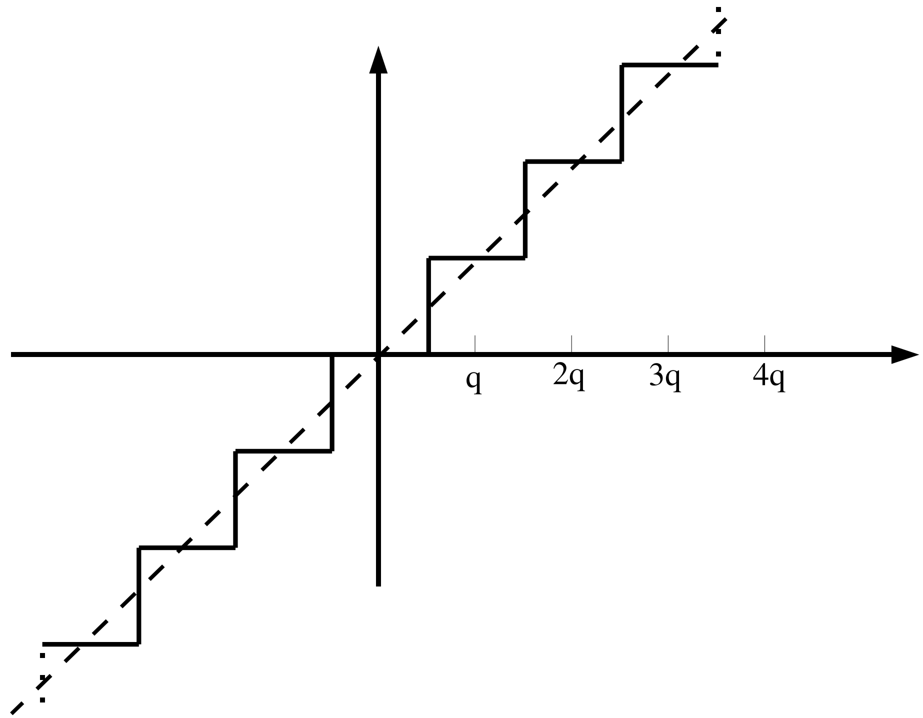
- There are different quantisation strategies:
  - truncation: ignore the least significant bit of the fractional part.
  - round-off: ignore the least signficant bit if the first bit lost is 0, otherwise add 2^{-l}
- Quantisation introduces an error: $x = x_q + \epsilon$

Truncation



- Truncation introduces an error of ϵ with $\epsilon \leq q = 2^{-l}$

Round-off



- Round-off introduces an error of ϵ with $\epsilon \leq q/2$, with $q = 2^{-l}$

Worst case analysis (Beltram (1958))

- Let $H(z)$ be a controller, and let $H_1(z)$ be the transfer function between the quantisation point and the output.
- It is possible to write:

$$
\begin{aligned}
Y(z) &= H(z)U(z) && (1)\\
\hat{Y}(z) &= H(z)U(z) - H_1(z)E_1(z,x) && (2)\\
Y - \hat{Y} = \tilde{Y} &= H_1(z)E_1(z,x) && (3)
\end{aligned}
$$

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- $\tilde{y}(n) = \sum_0^n h_1(k)\epsilon_1(n-k;x)$

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- Not time invariant (Dependence on x)
- It is possible to build an upper bound:

$$
\begin{aligned}
|\tilde{y}(n)| &= |\textstyle\sum_0^n h_1\epsilon_1| \\
&\leq \textstyle\sum_0^n |h_1\epsilon_1| \\
&\leq \textstyle\sum_0^n |h_1||\epsilon_1| \\
&\leq \textstyle\sum_0^n |h_1|\frac{q}{2}
\end{aligned}
$$

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- Example:

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- We get:

$$\begin{aligned} \hat{y}(k+1) &= \alpha \hat{y}(k) + \epsilon(k) + u(k) \\ \tilde{y}(k+1) &= \alpha \tilde{y}(k) - \epsilon(k) \end{aligned}$$

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- ...and (for $\alpha < 1$ and for $k \to \infty$)

$$\tilde{y}(k) \leq \frac{q}{2}\sum_{0}^{\infty}|\alpha|^k \leq \frac{q}{2}\frac{1}{1-|\alpha|}$$

Steady state analysis

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- Then $\tilde{y}(\infty) = \sum_0^\infty h_1(n)\epsilon_s$
- Proceeding as above

$$\begin{aligned} |\tilde{y}(\infty)| &\leq \left|\sum_0^\infty h_1(n)\right| \frac{q}{2} \\ &= |H_1(1)|\frac{q}{2} \end{aligned}$$

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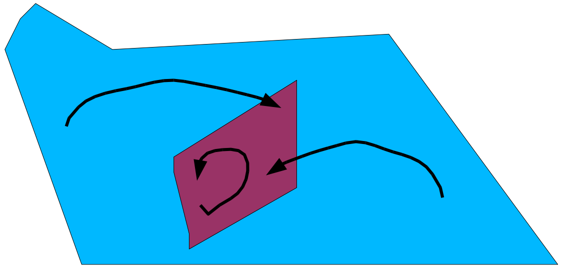
- however, it is possible to asymptotically restrict the state's evolution in a small region (controlled invariant) and....
- make it attractive from a larger set

Some definitions



- The set $D \subseteq \mathbb{R}^N$ is said controlled invariant for the single input systems $x(k+1) = Ax(k) + bu(k)$, with $u(k) \in \mathcal{U}$, if for all $x(k) \in D$ there exist u such that $x(k+1) \in D$

- Let $D \subseteq \Omega \subseteq \mathbb{R}^n$ with D controlled invariant. The set D is said $\Omega-$attractive from iff for all $x \in \Omega$ there exist a sequence of control values that steers x into D. If, moreover, this sequence is long at most H_p steps then D is said Ω-attractive in H_p steps.

A fundamental result

- Theorem (Picasso and Bicchi-2002) Let A, b be in control canonical form and α_i be the coeeficients of the characteristic polynomial.Assume that $u \in \mathcal{U} = \epsilon\mathbb{Z}$ and $Q_n(a) \subseteq \mathbb{R}$ be the hypercube of side a centered in the origin. Then the following facts hold:
  - $\forall\Delta \geq \epsilon, Q_n(\Delta)$ is controlled invariant
  - $\forall H_p \geq 0$, $\forall\Delta \geq 0$ $Q_n(\epsilon)$ is $Q_n(\Delta)$-attractive in H_p steps
  - $Q_n(\epsilon)$ is minimal in the following sense:
    1. $x(k) \in Q_n(\epsilon)$ $\exists! u \in \mathcal{U}$ s.t. $x(k+1) \in Q_n(\epsilon)$
    2. if $\sum_i |\alpha_1| > 1$ then $\forall\delta < \epsilon$, $Q_n(\delta)$ is not controlled invariant.

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- The controlled invariance of the interval can be imposed as follows:

$$x(k+1) \in [-\Delta/2, \Delta/2] \forall x(k) \in [-\Delta/2, \Delta/2] \leftrightarrow$$

$$\forall x(k) \in [-\Delta/2, \Delta/2] \exists u \in \mathcal{U} s.t. - \Delta/2 \leq ax(k) + u(k) \leq \Delta/$$

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- The segment of acceptable $u(k)$ is Δ, so the quantiser grain has to be $\epsilon \leq \Delta$

Simplified proof - I

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- The only solution in $\epsilon\mathbb{Z}$ s.t.
  $x(k+1) \in [-\epsilon/2,\, \epsilon/2]$ is $u(x) = \lfloor \frac{-ax+\epsilon/2}{\epsilon} \rfloor \epsilon$

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- if $\Delta = \epsilon$ then it is required $-\epsilon/2 - ax(k) \leq u(k) \leq \epsilon/2 - ax(k)$
- The only solution in $\epsilon\mathbb{Z}$ s.t. $x(k+1) \in [-\epsilon/2, \, \epsilon/2]$ is $u(x) = \lfloor \frac{-ax+\epsilon/2}{\epsilon} \rfloor \epsilon$
- the feedback law above is a quantised deadbeat!

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- So far we have seen the problem of quantization referred to Input/output data
- The problem is there also for parametric uncertainties inthe controller
- For example suppose you choose a controller given by: $y(k+1) = \alpha y(k) + u(k)$
- Due to finite precision math the system we will deal with is:
  $y(k+1) = (\alpha + \delta\alpha)y(k) + u(k) + \epsilon(k)$

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Parameters round-off - (continued)

- In the example above the $Z-$transform that we get is $z - \alpha + \delta\alpha$
- If we want to place a pole at 0.995 we need at least a precision of three digits: $\delta\alpha \leq 0.005$
- Robustness analysis can be conducted using the root-locus-Jury critterion methods
- We will show a different approach based on linearised sensitivy analysis

Linearised sensitivity analysis

- Let the closed loop characteristic polynomial be:
$$P = z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n$$
- Let the roots of the system be: $\lambda_i = r_i e^{j\theta_i}$
- Suppose α_k is subject to uncertainity and that we are interested in evaluating variations of λ_j, that we assume to be a *single* root of P
- The Polynomial P can be thought of as a function of z and α_k: $P(z, \alpha_k)$

Lin. sensitivity an. - (continued)

- First order series expansion around $z = \lambda_j$:
  $P(\lambda_j + \delta\lambda_j, \alpha_k + \delta\lambda_k) \approx P(\lambda_j, \alpha_k) + \left.\frac{\partial P}{\partial z}\right|_{z=\lambda_j} \delta\lambda_j + \frac{\partial P}{\partial \alpha_k}\delta\alpha_k + \ldots$

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- Observing that $P(\lambda_j, \alpha_k) = 0$ we get:

$$\delta\lambda_j \approx \left.\frac{\partial P/\partial \alpha_k}{\partial P/\partial z}\right|_{z=\lambda_j} \delta\alpha_k$$

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- Writing $P(z, \alpha_k) = (z - \lambda_1)\dots(z - \lambda_n)$, we get for the denominator:

$$\frac{\partial P}{\partial z}\Bigg|_{z=\lambda_j} = \prod_{l \neq j}(\lambda_j - \lambda_l)$$

Lin. sensitivity an. - (continued)

- ...hence,

$$\delta\lambda_j = -\frac{\lambda_j^{n-k}}{\prod_{l\neq j}(\lambda_j - \lambda_l)}\delta\alpha_k$$

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$$\delta \lambda_j = -\frac{\lambda_j^{n-k}}{\prod_{l \neq j} (\lambda_j - \lambda_l)} \delta \alpha_k$$

- Consideration 1: assuming a stable λ_j the highest sensitivity is for $n = k$ i.e. for the variations of α_n
- Consideration 2: Looking at the denominator roots should be far apart: clusters empahsise senistivity

Example

- Suppose we need to implement a low pass filter with sharp cutoff
  - we need a cluster of roots $\longrightarrow$ high sensitivity for canonical implementations
  - it is much more convenient to use a series or parallel composition of low order filters (sensitivity is on one factor).