Ingegneria dell 'Automazione - Sistemi in Tempo Reale

Selected topics on Hybrid Systems

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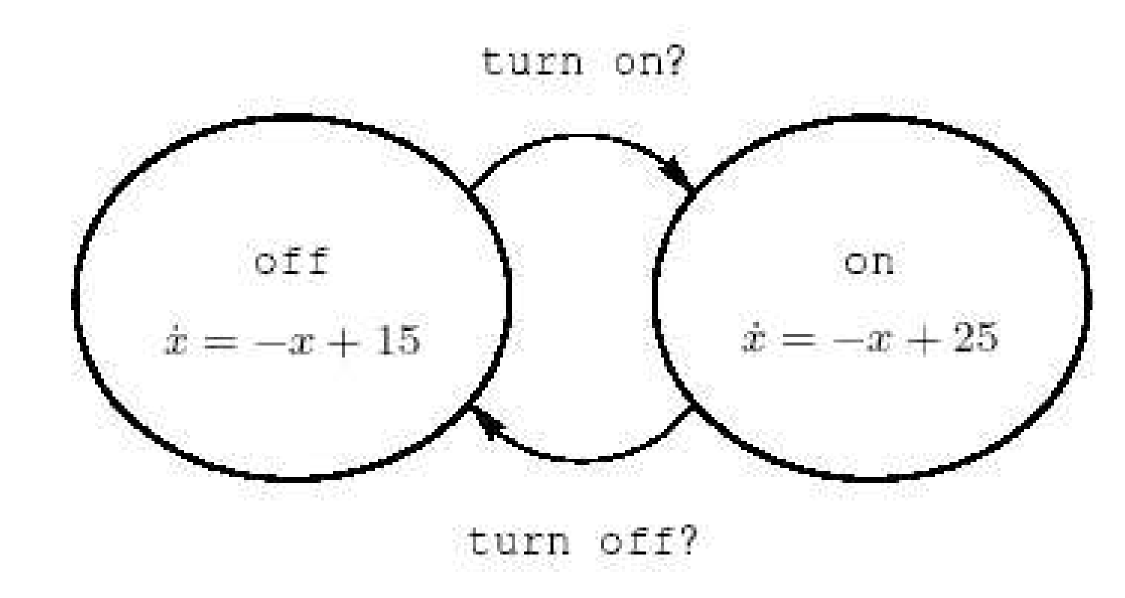
Modeling

Hybrid Automaton

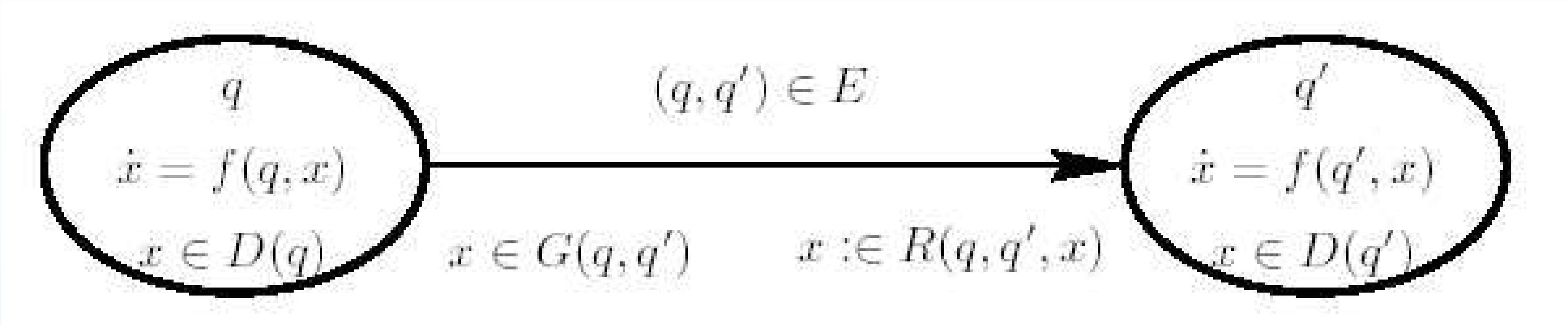
- A **hybrid automaton** is a formal model of a hybrid system
- It defines the evolution of the hybrid system state

Example

Thermostat controlling a room temperature T



Hybrid Automaton

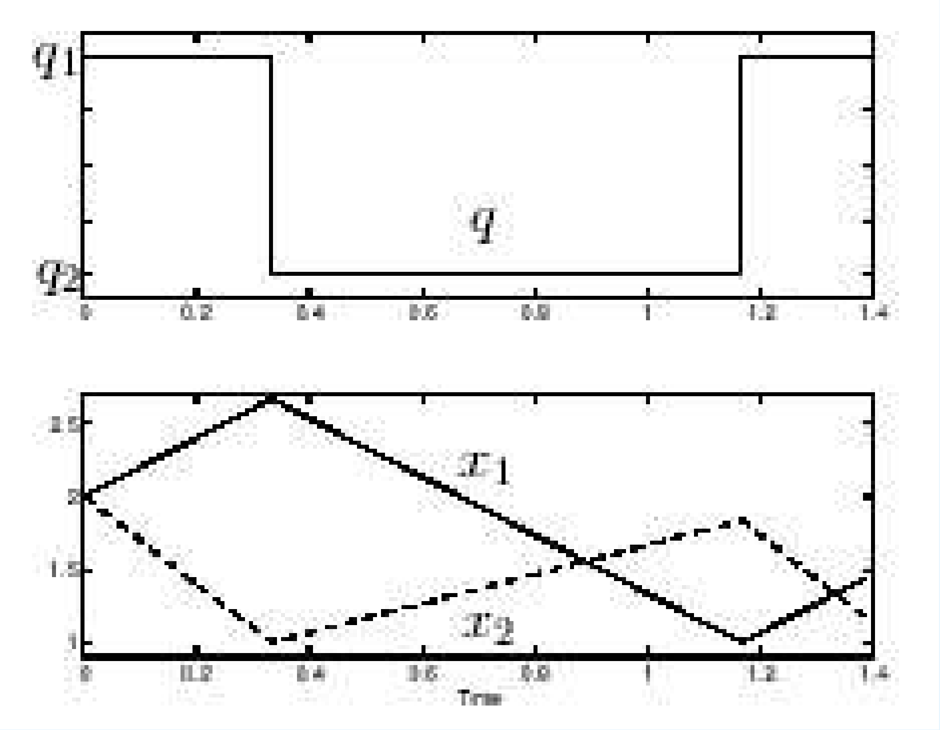


- Defined by $H = (Q, X, \mathrm{Init}, f, D, E, G, R)$
  - Q discrete state space and X continuous state space
  - $\mathrm{Init} \subseteq Q \times X$ initial states
  - $f : Q \times X \to X$ vector fields
  - $D : Q \to 2^X$ domains
  - $E \subset Q \times Q$ edges
  - $G : E \to 2^X$ guards
  - $R : E \times X \to 2^X$ resets

Solution of Hybrid Automaton

A **solution** $\chi = (\tau, q, x)$ of H consists of

- **Time trajectory** τ: time line on which the solution is defined
- **State trajectory** (q, x): state evolution (defined on τ) of the hybrid automaton



Time Trajectory τ

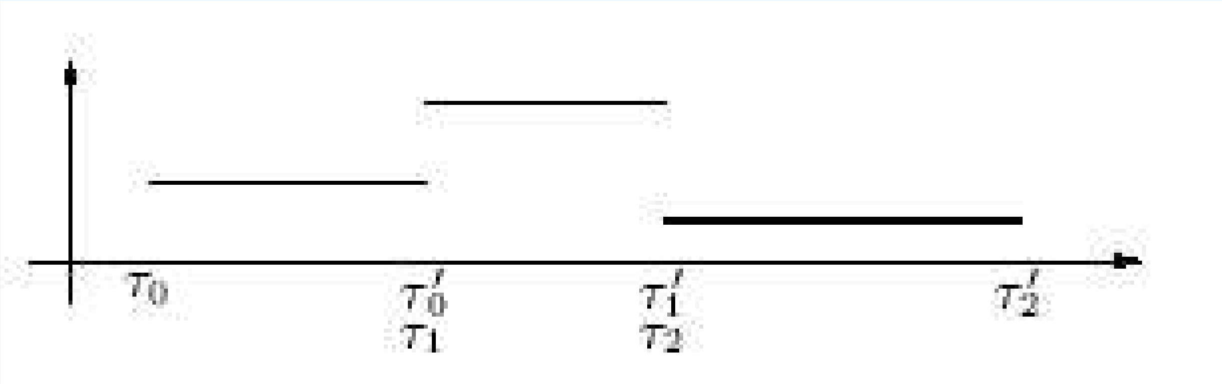
A sequence of (time) intervals

$$\tau = \{I_i\}_{i=0}^{N}$$

such that

- $I_i = [\tau_i, \tau_i']$ for all $i < N$;
- if $N < \infty$, then either $I_N = [\tau_N, \tau_N']$, or $I_N = [\tau_N, \tau_N')$; and
- $\tau_i \leq \tau_i' = \tau_{i+1}$ for all i.

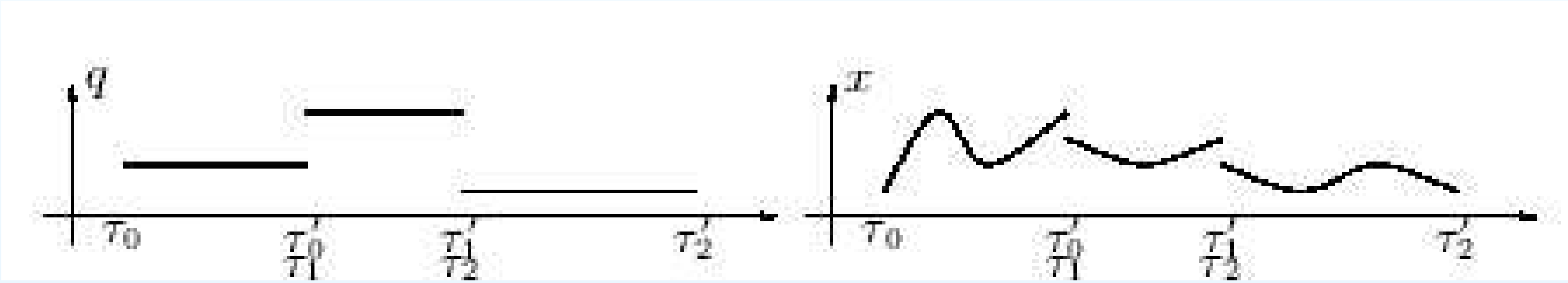
Notation: $\langle\tau\rangle = \{0, 1, \ldots, N\}$



Solution $\chi = (\tau, q, x)$

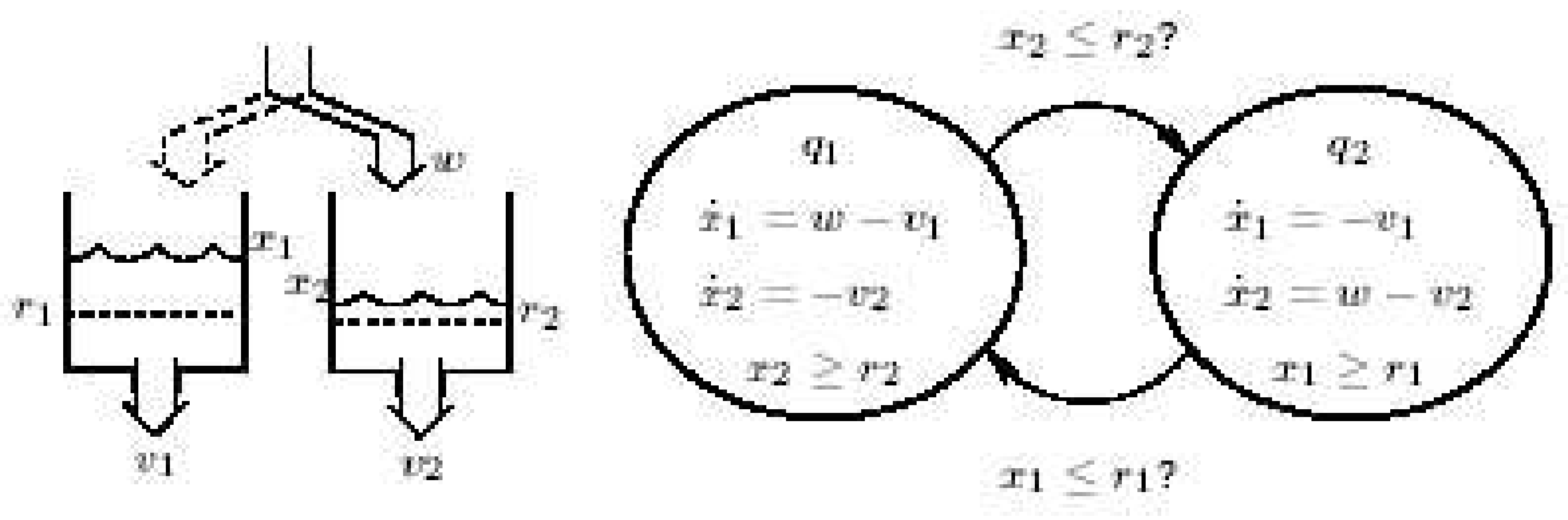
$\tau = \{I_i\}_{i=0}^{N}$, $q : \langle\tau\rangle \to Q$, $x = \{x^i : i \in \langle\tau\rangle\}$, $x^i : I_i \to X$ such that

- **Initialization:** $(q(0), x^0(0)) \in \mathrm{Init}$,
- **Time-driven:** for all $t \in [\tau_i, \tau_i')$, $\dot{x}^i(t) = f\big(q(i), x^i(t)\big)$ and $x^i(t) \in D(q(i))$
- **Event-driven:** for all $i \in \langle\tau\rangle \setminus \{N\}$, $e = (q(i), q(i+1)) \in E$, $x^i(\tau_i') \in G(e)$, and $x^{i+1}(\tau_{i+1}) \in R(e, x^i(\tau_i'))$

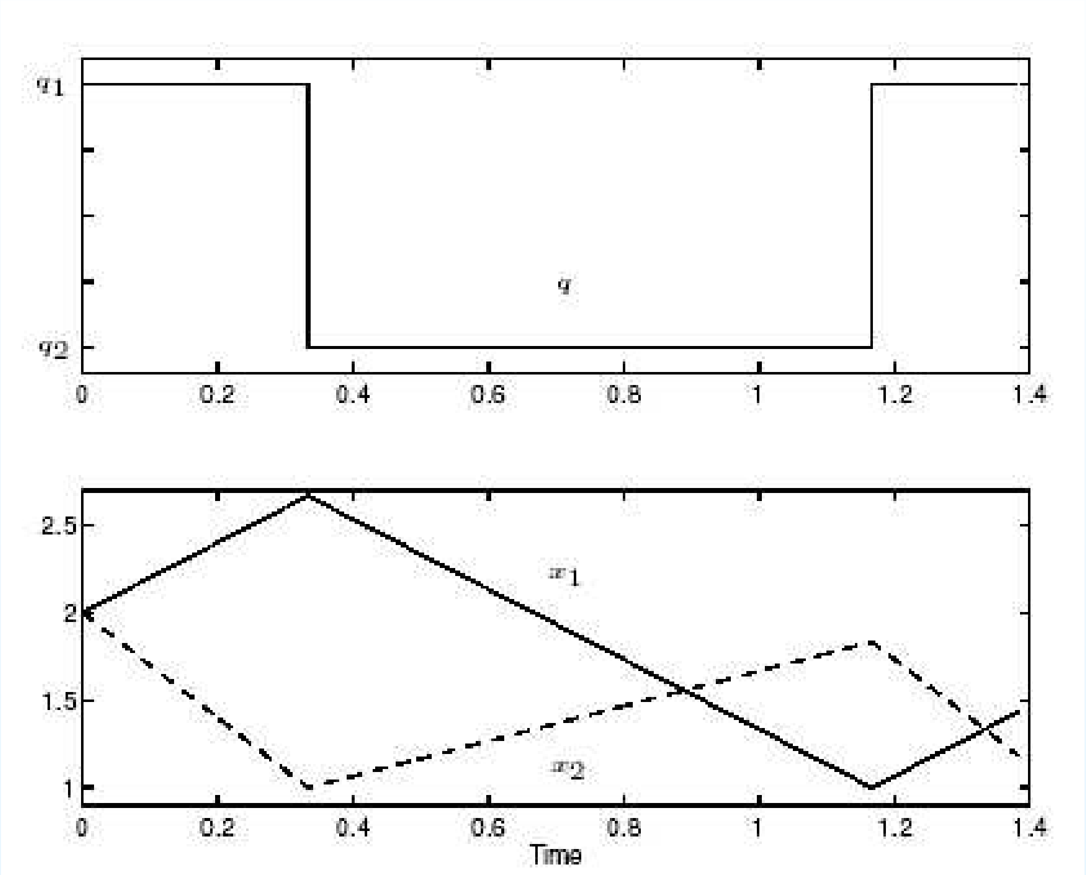


Example 1: Water Tank System

Control objective is to keep water volumes above r_1 and r_2 by switching the inflow



Example 1: trajectories



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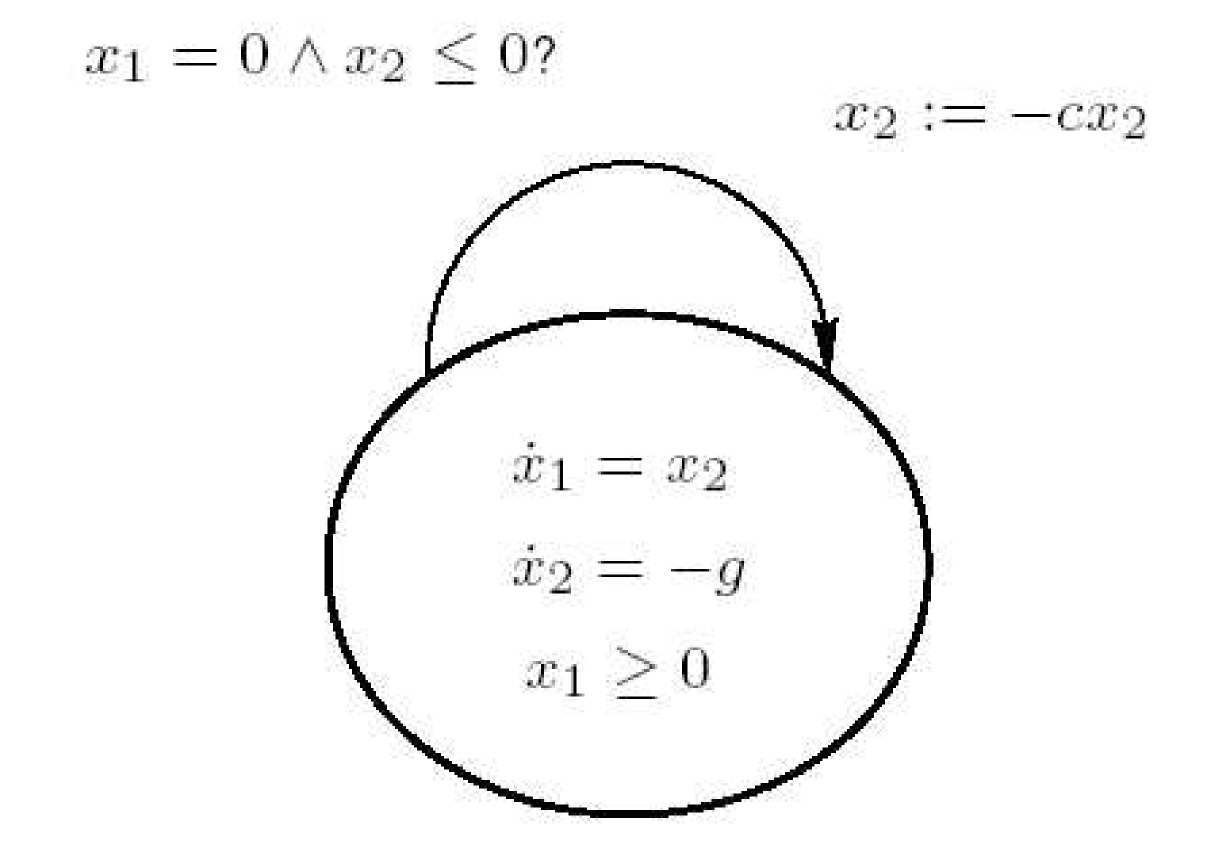
- $R(q_1, q_2, x) = R(q_2, q_1, x) = \{x\}$.

Example 2: Bouncing ball

- A ball that loses a fraction of its energy at each bounce



Example: Differential Equation and Automaton

- A continuous-time system $\dot{x} = f(x)$ is a hybrid automaton with a single discrete state
- A finite automaton is a hybrid automaton with no continuous dynamics

Timed Automaton

A timed automaton is a hybrid automaton $H = (Q, X, \text{Init}, f, D, E, G, R)$ with

- $Q = \{q_1, \ldots, q_m\}$;
- $X = \mathbb{R}^n_+$;
- $\text{Init} \subseteq Q \times X$
- $f(q, x) = (1, \ldots, 1)^T$
- $D(q) = [a_1, b_1] \times \ldots [a_m, b_m]$
- $E \subset Q \times Q$
- $G(e) = [c_1, d_1] \times \ldots [c_m, d_m]$
- $R(e, x) = 0$

A timed automaton models n clocks (e.g., computation time)

Properties

Properties of Hybrid Systems

Liveness For all $(q_0, x_0) \in \mathrm{Init}$, there exists at least one (infinite) solution from (q_0, x_0)

Determinism For all $(q_0, x_0) \in \mathrm{Init}$, there exists at most one solution starting from (q_0, x_0)

Zenoness Finite execution time $\tau_\infty = \sum_{i=1}^{\infty} (\tau_i' - \tau_i) < \infty$

Stability Stability of equilibria and other invariant sets

Reachability Reachable states $\mathrm{Reach} \subset Q \times X$

The case of ODE

- Consider a differential equation:

$$\dot{x} = f(x, v)$$

  where:
  - $x \in \mathbb{R}^n$ is the state space
  - $v = (u,\, d) \in U \times D$ is the input vector (u is the command variable and d is an external disturbance term
  - f is a vector field
  - the initial state $x(0) \in Init$ where $Init \subseteq X$

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous if there exist $\lambda > 0$ s.t.:

$$||f(x_1) - f(x_2)|| \leq \lambda ||x_1 - x_2||, \forall x_1, x_2 \in \mathbb{R}^n$$

Fundamental theorems

- **Theorem: Local Existence and Uniqueness**: if $f(x, v)$ is piecewise continuous in v, and Lipschitz continuous in x for $x \in B_r(x_0)$ with $r > 0$ ($B_r(x_0) = \{x|||x - x_0|| \leq r$ and $||f(x_0)|| \leq h$ for $h > 0$). Then there exists $\delta > 0$ s.t. the system has a unique solution in $[0, \delta]$
-
- **Theorem: global Existence and Uniqueness**: if $f(x, v)$ is piecewise continuous in v, and Lipschitz continuous in x . Then there exists $\delta > 0$ s.t. the system has a unique solution in $\forall t \in \mathbb{R}$. Moreover solutions have a contiuous dependence on the intial conditions.

Interpretation

The fundamental Theorems shown earlier can be interpreted as follows:

- Local existence of solutions: *the system is nonblocking*
- Uniqueness of solutions: *the system is deterministic*
- Non-Zenoness: *solutions can be extended over arbitrarily long time horizons*

Liveness

Definition

For all $(q_0, x_0) \in \mathrm{Init}$, there exists **at least one** (infinite) solution from (q_0, x_0)

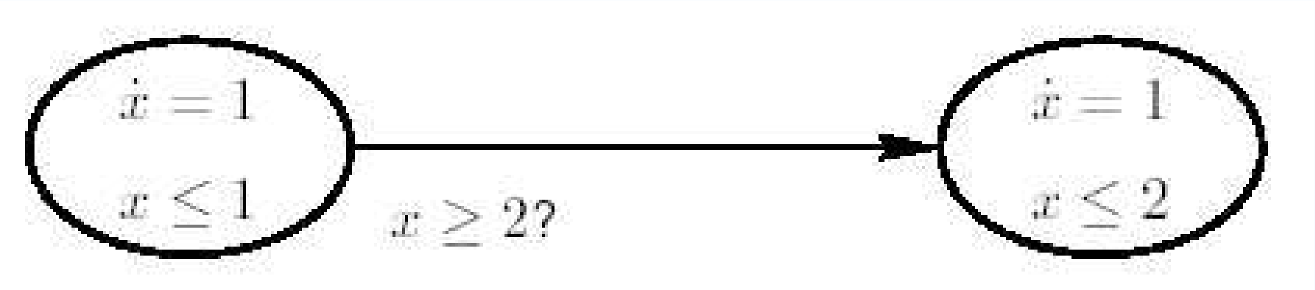
Theorem

A hybrid automaton is live if for all reachable states for which continuous evolution is impossible, a discrete transition is possible

- It is reasonable to expect that models for physical systems should be live
- If a hybrid automaton is not live, it can be due to over-simplifications in the model

Example

Let $\text{Init} = (q_1, 0)$. Then the following hybrid automaton is not live (blocking):



Determinism

Definition

For all $(q_0, x_0) \in \mathrm{Init}$, there exists **at most one** solution starting from (q_0, x_0)

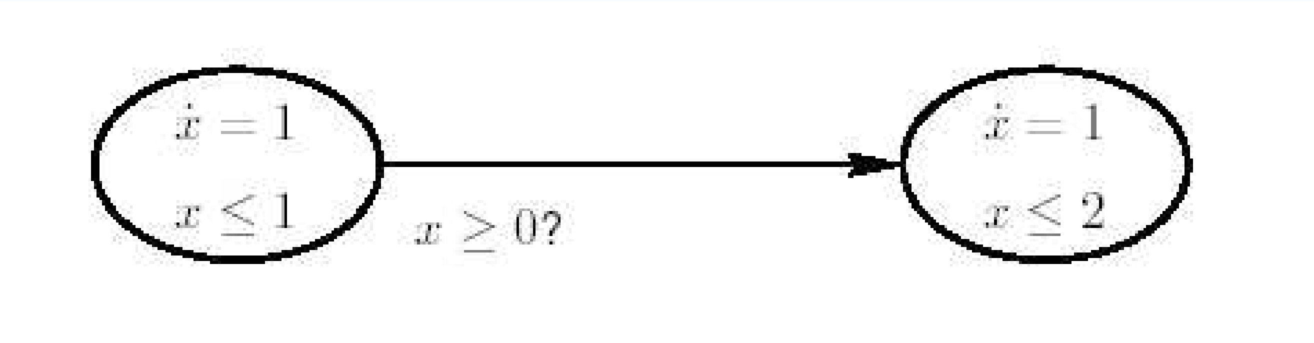
Theorem

A hybrid automaton is deterministic if there is

- no choice between continuous evolution and a discrete transition, and
- a discrete transition can never lead to multiple destinations

Example

Let $\mathrm{Init} = (q_1, 0)$. Then the following hybrid automaton is non-deterministic:

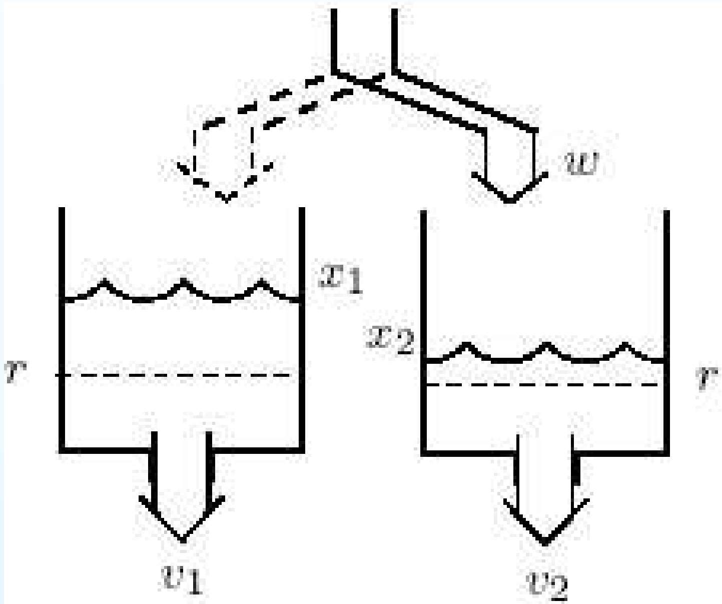


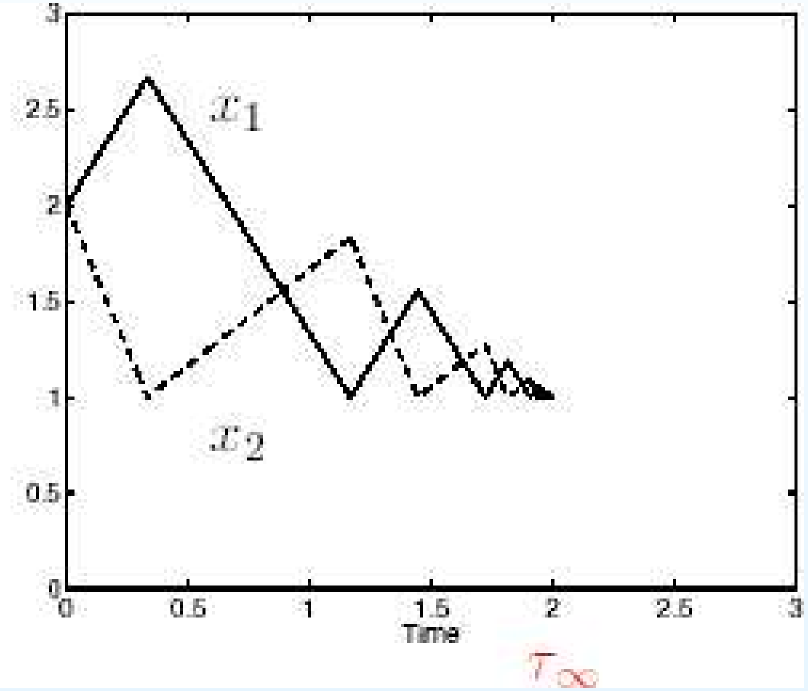
Zeno Solution of Hybrid Automaton

A solution $\chi = (\tau, q, x)$ is Zeno if $\tau_\infty = \sum_{i=1}^{\infty} (\tau_i' - \tau_i) < \infty$

Example—Water tank system: If $\max\{v_1, v_2\} < w < v_1 + v_2$ then

$\tau_\infty = (x_1(0) + x_2(0) - 2r)/(v_1 + v_2 - w) < \infty$





Execution is not defined for $t \geq \tau_\infty$

Zeno of Elea (490–430 B.C.)

- Born in southern Italy
- Met Socrates in Athens 449 B.C.
- Went back to Elea
  and into politics
- Tortured to death

- Paradoxes “proved” that motion and time are illusions
- Led to mathematical problems not solved until 19th century

Zeno

- A solution is Zeno if it exhibits infinitely many discrete jumps in finite time
- Zeno is a truly hybrid phenomenon: it cannot be formulated for a purely discrete system without the notion of continuous time
- Zeno is due to that the model does not reflect reality with sufficient detail
- A hybrid automaton have Zeno solutions only if (Q, E) is a cyclic graph (a graph with a loop)

Example—Bouncing ball

For which values of c is the solutions of the bouncing ball hybrid automaton Zeno?

Regularization

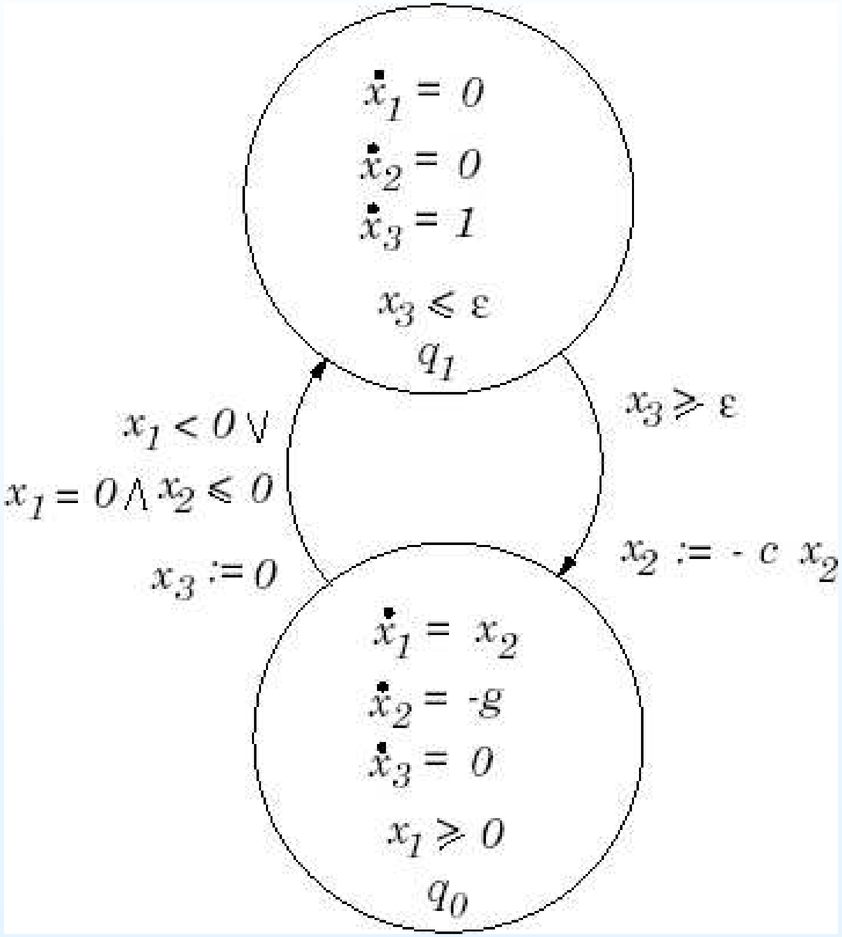
- Regularizatin of a differential equation corresponds to introducing a slight modification to the equation so as to eliminate a singularity point
- The same is in principle possible also for hybrid systems
- Given a Zeno automata H, we replace it by an automata H_ϵ (parametrized by ϵ), whose trajectories converges to H when $\epsilon \longrightarrow 0$
- For instance, in the case of the double tank, the Zeno behaviour is due to a modeling oversimplification: the inflow is modeled as ideal swithc
- Things would be different if we introduced the dynamics of the outlet

Example

- consider the case of the bouncing ball
- the problem is the instantaneous inversion of the velocity
- A possible regularization is to introduce a delay for the bouncing phase
- How would you do that? (hint model time as a clock)

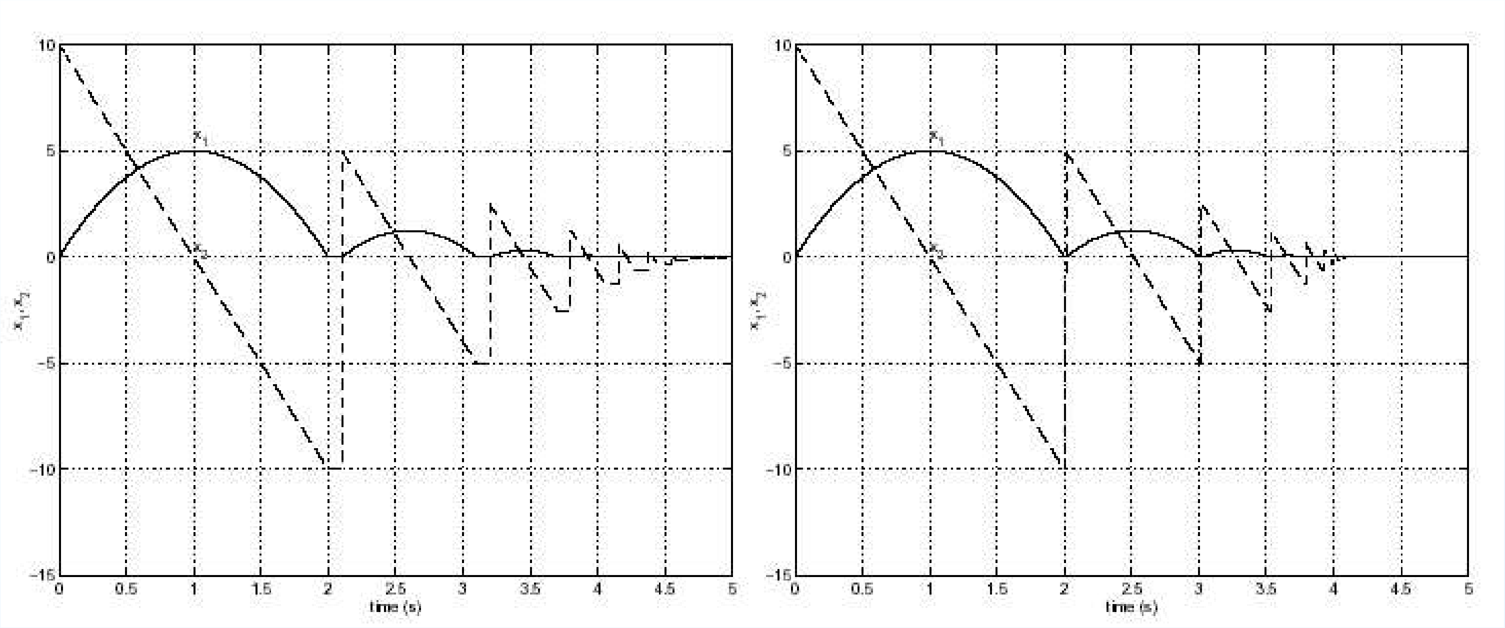
Example - I

Regularized automaton



Example - II

Trajectories

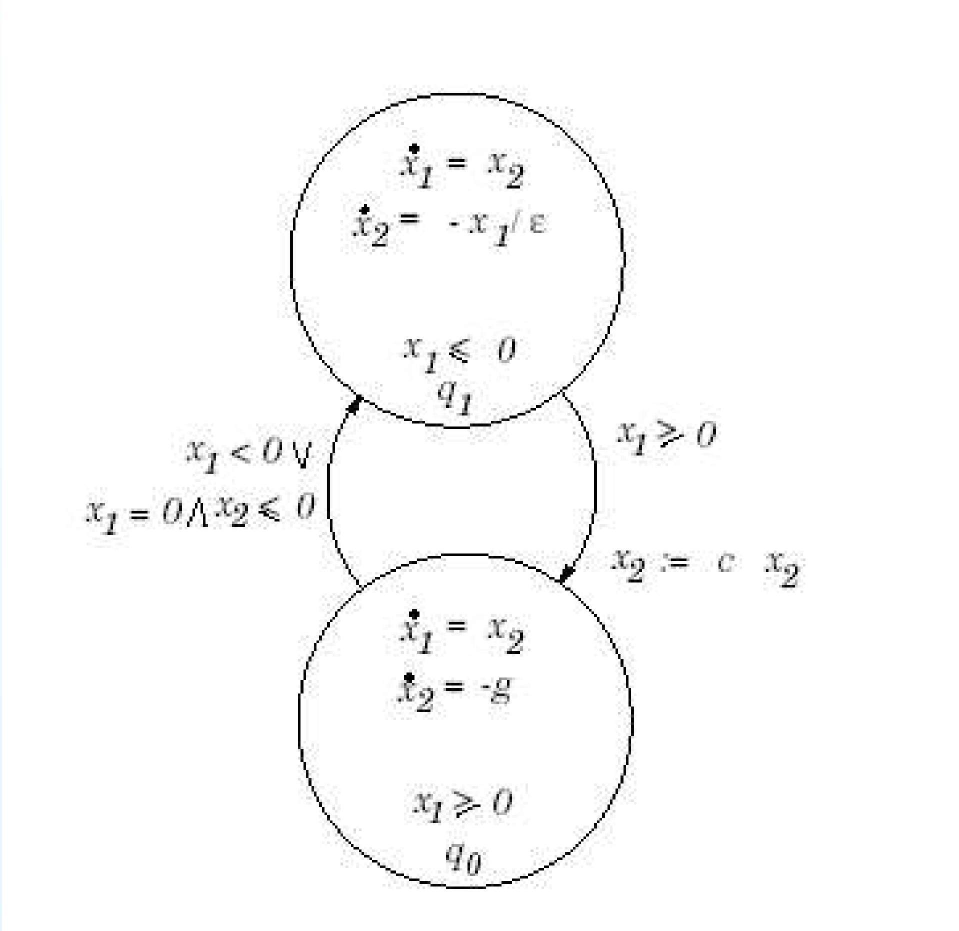


Example - IV

- Another approach is to introduce an eleastic force (i.e. model the ground as a spring)
- Why don't you try?

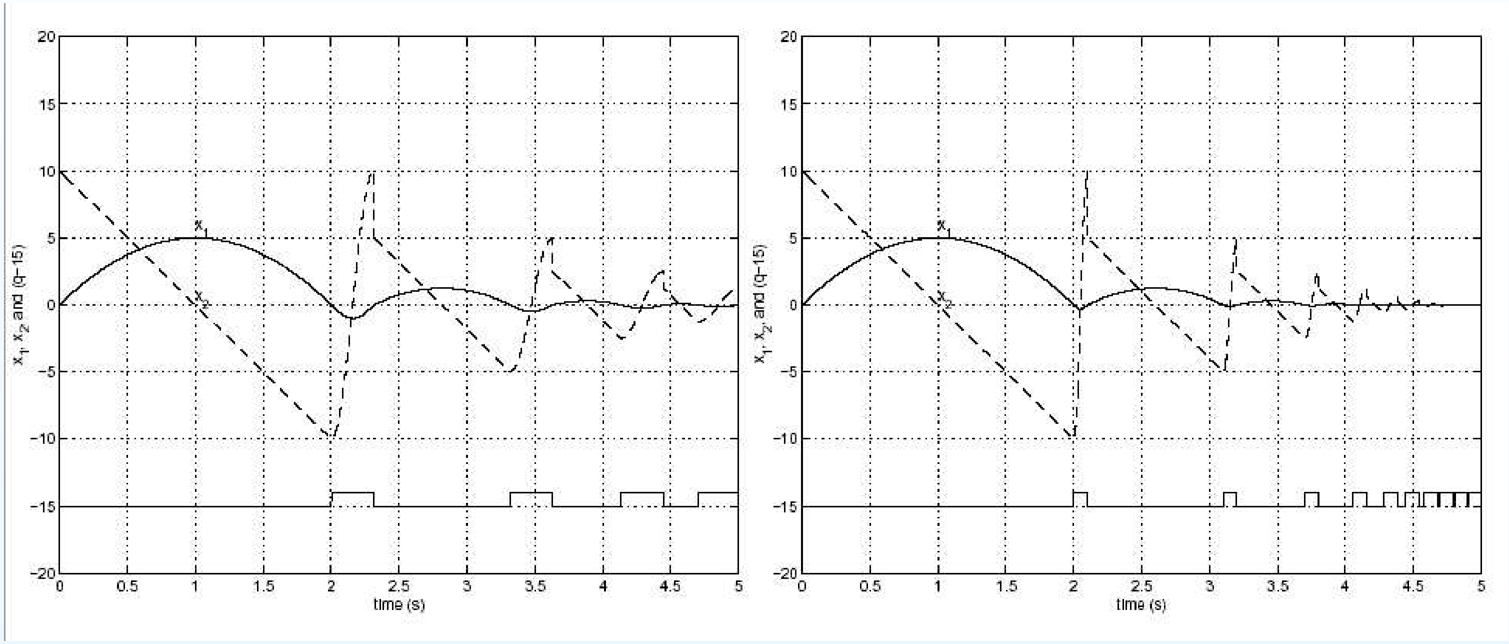
Example - V

Regularized automaton



Example - V

Trajectories



Switched systems

Switched Systems

Let Ω_q, $q = 1, \dots, m$ denote a partition of the continuous state space $\mathbb{R}^n$
A **switched system** is then defined as

$$\dot{x} = f_q(x), \qquad x \in \Omega_q$$

Example
$x \in \mathbb{R}^2$, Ω_q quadrant q, $q = 1, \dots, 4$, and

$$\dot{x} = A_q(x)$$
$$x \in \Omega_q$$

Switched systems

- $q(t)$ is a piecewise constant signal
- at switching times: $x = \rho(q,\, q^-,\, x^-)$
- A solution is a pair $(x(t), q(t))$ for which
  - on every open time interval for which $q(t)$ is constant, we get:

$$\dot{x} = f_q(x)$$

  - upon switching times we get:

$$x(t) = \rho(q(t), q^-(t), x^-(t))$$

Switched System as Hybrid Automaton

$$\dot{x} = f_q(x), \qquad x \in \Omega_q$$

corresponds to the hybrid automaton

- $Q = \{1, \ldots, m\}$, $X = \mathbb{R}^n$, $\mathrm{Init} \subset \{q\} \times \Omega_q$
- $f(q, x) = f_q(x)$
- $D(q) = \Omega_q$
- $(q, q') \in E$ if $D(q)$ to $D(q')$ are "neighbors" (i.e., $\overline{D(q)} \cap \overline{D(q')} \neq \emptyset$) and there are solutions that go from $D(q)$ to $D(q')$
- $G(q, q') = \overline{D(q)} \cap \overline{D(q')}$
- $R(q, q', x) = x$

Stability

Stability

A solution x^* of switched system is stable if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all solutions x

$$\|x(0) - x^*(0)\| < \delta \quad \Rightarrow \quad \|x(t) - x^*(t)\| < \epsilon, \quad \forall t > 0$$

- Similar stability definition as for continuous systems
- Can be generalized to hybrid automata

Lyapunov's Second Method

Let $x^* = 0$ be an equilibrium point of $\dot{x} = f(x)$. If there exists a function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$
\begin{aligned}
&V(0) = 0 \\
&V(x) > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\} \\
&\dot{V}(x) \leq 0, \quad \forall x \in \mathbb{R}^n,
\end{aligned}
$$

then x^* is stable

Lyapunov Function for Linear System

$\mathrm{Re}\,\lambda_i(A) < 0$ for all i if and only if for every positive definite $Q = Q^T$ there exists a positive definite $P = P^T$ such that

$$PA + A^T P = -Q$$

A Lyapunov function for a linear system

$$\dot{x} = Ax$$

is given by

$$V(x) = x^T P x$$

In particular,

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x < 0$$

Example

$$\dot{x} = A_1 x = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix} x$$

Then,

$$P = [\texttt{lyap} \text{ in Matlab}] = \begin{pmatrix} 0.2752 & -0.0225 \\ -0.0225 & 2.7478 \end{pmatrix}$$

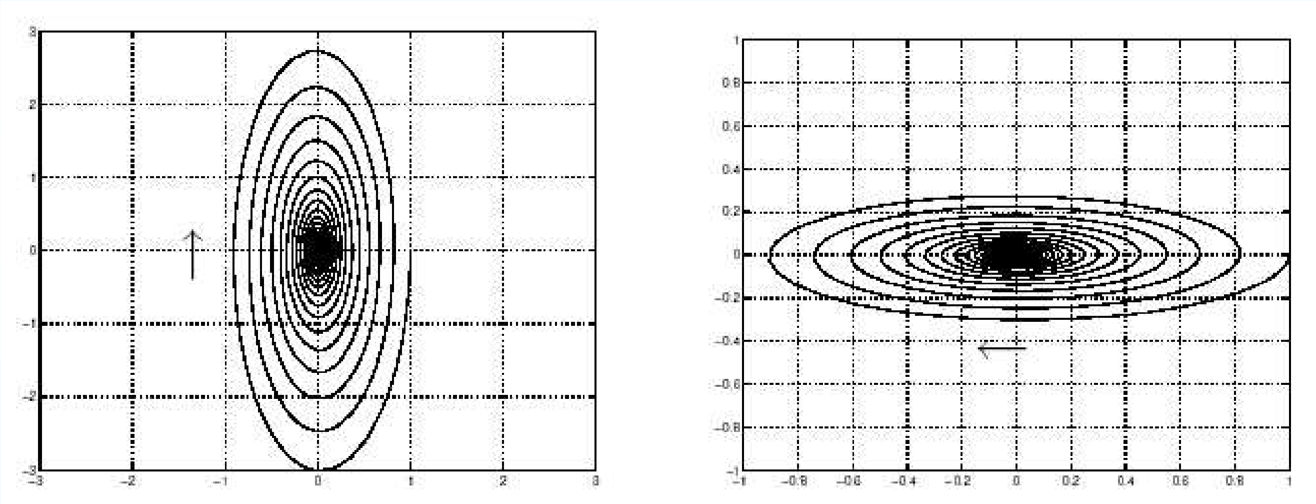
solves the Lyapunov equation $A_1 P + P A_1^T = -I$. Then, $V = x^T P x$ fulfills the three conditions in the Lyapunov theorem (check!). Hence, $x^* = 0$ is stable.

Note that $\lambda(A_1) = -1 \pm i10\sqrt{10}$

Example

$$\dot{x} = A_1 x = \begin{pmatrix} -1 & 10 \\ -100 & -1 \end{pmatrix} x, \qquad \dot{x} = A_2 x = \begin{pmatrix} -1 & 100 \\ -10 & -1 \end{pmatrix} x$$

have the following phase portraits:

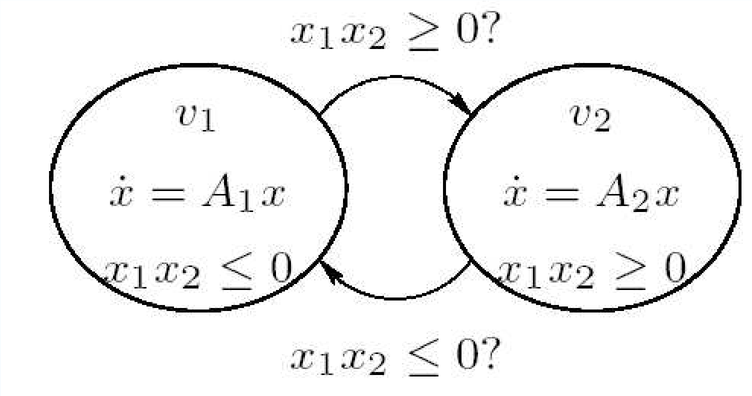


Note

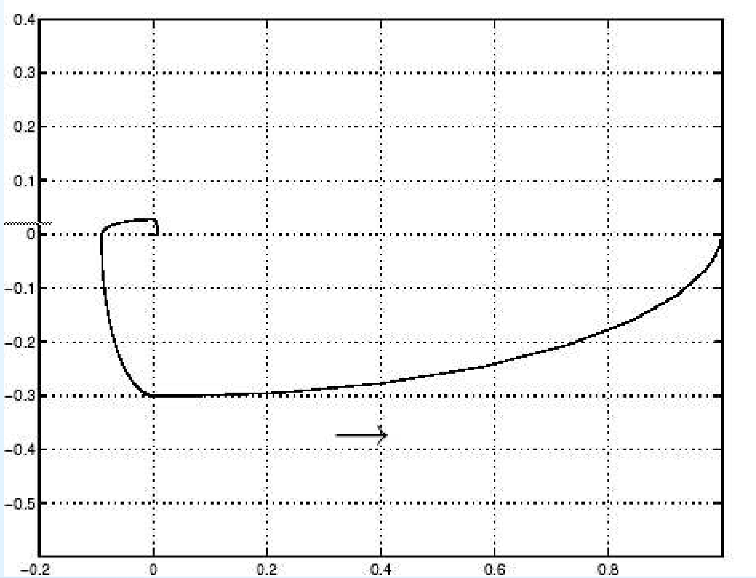
- Both systems are stable with $\lambda(A_i) = -1 \pm i10\sqrt{10}$, $i = 1, 2$
- Lyapunov function $V(x(t))$ is decreasing along the solution

Example: Stable+Stable=Unstable

Consider switched system corresponding to hybrid automaton:

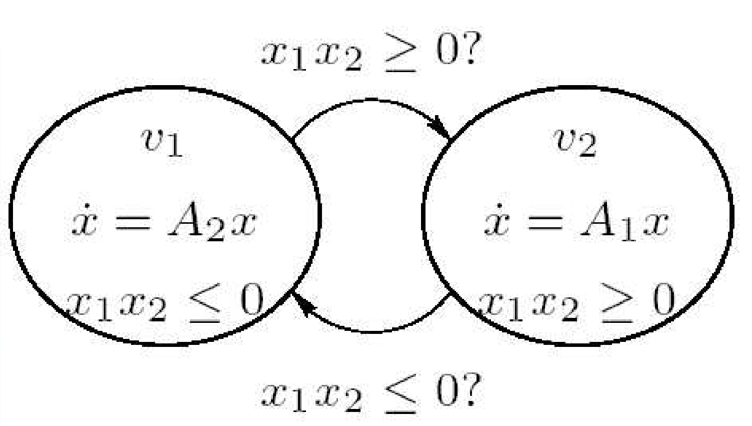


Even if A_1 and A_2 are stable, the switched system is unstable:

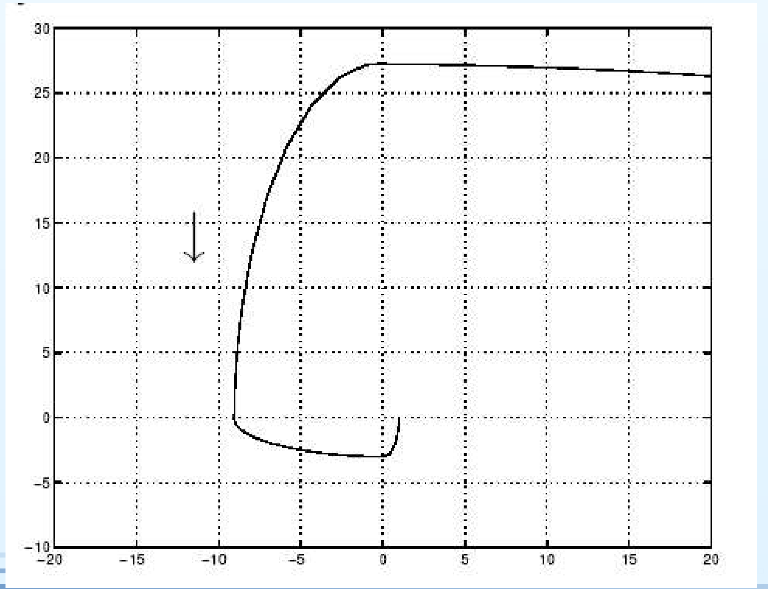


Example: Stable+Stable=Stable

Let A_1 and A_2 change place:



Then, also the switched system is stable:



Multiple Lyapunov Functions

Suppose x^* is an equilibrium of each mode $q = 1, \ldots, m$ of the switched system

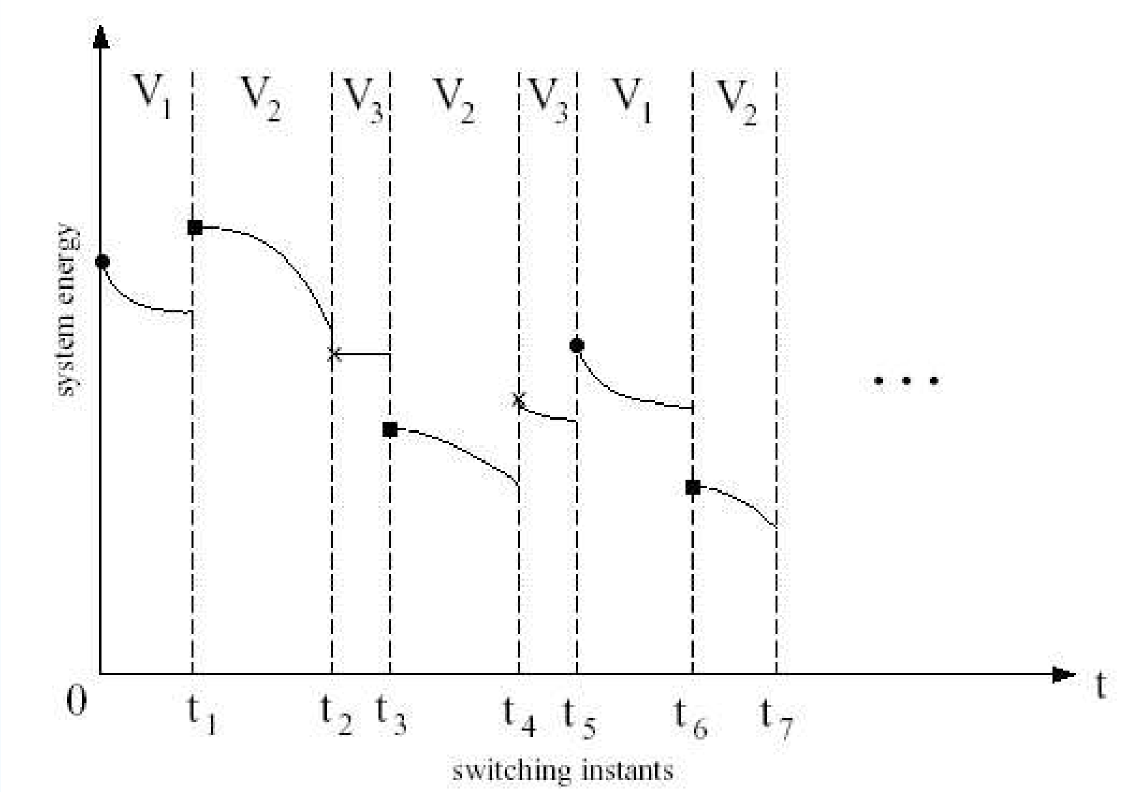
$$\dot{x} = f_q(x), \qquad x \in \Omega_q$$

If there exist functions $V_1, \ldots, V_m$ such that

$$V_q(0) = 0, \quad V_q(x) > 0, \ \forall x \in \mathbb{R}^n \setminus \{0\}$$
$$\dot{V}_q(x(t)) \leq 0, \quad \text{whenever } x(t) \in \Omega_q$$

and the sequences $\{V_q(x(\tau_{i_q}))\}$, $q = 1, \ldots, m$ are non-increasing, where τ_{i_q} are the time instances when mode q becomes active, then x^* is stable.

Example



Whenever the system enters into a mode its energy is lower than the last time it entered into the same mode.

Stability of switched systems under arbitray switching

- Assume that switching signal can be any piecewise constant signal

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- If one of the vector fields f_q is aymptotically unstable then also the switched sytem is. Why?

Stability of switched systems under arbitray switching

- Assume that switching signal can be any piecewise constant signal
- Assume that $\rho(q, q^-, x^-) = x^-$ (no resets)
- If one of the vector fields f_q is aymptotically unstable then also the switched sytem is. Why?
- Because as a special case we can stay forever in the unstable state

Commuting System Matrices

Consider the system $\dot{x} = A_\sigma x$, where $\sigma : [0, \infty) \to \{1, \dots, m\}$ is an arbitrary switching sequence. If all A_q are stable and

$$A_k A_\ell = A_\ell A_k, \qquad k, \ell \in \{1, \dots, m\}$$

then the origin is stable.

Proof for $m = 2$: If $A_1 A_2 = A_2 A_1$ then $\exp A_1 \exp A_2 = \exp A_2 \exp A_1$ (why?). Then, for time trajectory τ and $t \in [\tau_i, \tau_i']$,

$$\begin{aligned} x(t) &= \exp[A_1(t - \tau_i)] \exp[A_2(\tau_{i-1}' - \tau_{i-1})] \cdots \exp[A_1(\tau_0' - \tau_0)] x_0 \\ &= \exp\left[A_1[(t - \tau_i) + \cdots + (\tau_0' - \tau_0)]\right] \\ &\qquad \times \exp\left[A_2[(\tau_{i-1}' - \tau_{i-1}) + \cdots + (\tau_1' - \tau_1)]\right] x_0 \end{aligned}$$

Stability follows from that A_1 and A_2 are stable.

Common Lyapunov Function

Consider the system

$$\dot{x} = A_\sigma x$$

where $\sigma : [0, \infty) \to \{1, \dots, m\}$ is an arbitrary switching sequence. If there exists $P > 0$, such that

$$PA_q + A_q^T P < 0, \qquad q = 1, \dots, m$$

then the origin is stable

- $V(x) = x^T P x$ is a common Lyapunov function for all systems $\dot{x} = A_q x$

A simple consideration

- finding $P > 0$ s.t. $A_q^T P + PA_q < 0$ is a trivial problem by using any LMI solver
- If matrices A_q are all lower triangular or upper triangular and asymptotically stable then the system is asymptotically stable
- Why?
  - it is possible to find a diagonal common lyapunov function

Dwell Time

- Slow switching: switching with dwell time $\mathcal{S}_D[\tau_D]$ τ_D the interval between two mode changes is grater than or equal to τ_D

Theorem: if A_q are all asymptotically stable, there exists a dwell-tim τ_D s.t. the switched system is A.S.

A Stabilizing Switching Sequence

Suppose there exist $\mu_q \geq 0$, $q \in Q$ and $\sum_{q=1}^{m} \mu_k = 1$, such that $A = \sum_{q=1}^{m} \mu_k A_k$ is stable. Then, a stabilizing switching sequence $\sigma : [0, \infty) \rightarrow Q := \{1, \ldots, m\}$ for

$$\dot{x} = A_\sigma x,$$

is given by

$$\sigma(x) = \arg \min_{q \in Q} x^T (A_q^T P + P A_q) x$$

where $P > 0$ is the solution to $A^T P + P A = -I$.

Proof: Follows from that $\sum_{q=1}^{m} \mu_q z^T (A_q^T P + P A_q) z < 0$ and $\mu_q \geq 0$, which gives $x^T (A_q^T P + P A_q) x < 0$ for any x.