Signal and Systems

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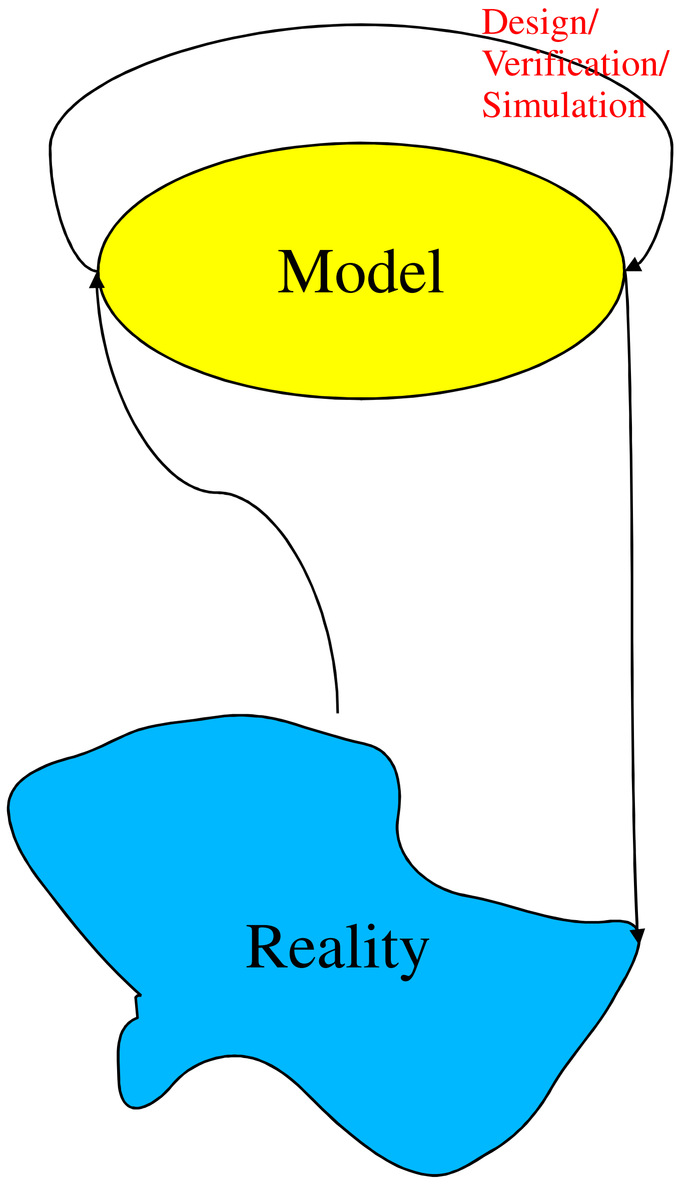
Signal and Systems

- Signal and systems are useful *models* to
  - produce
  - manipulate
  - transmit
  - store

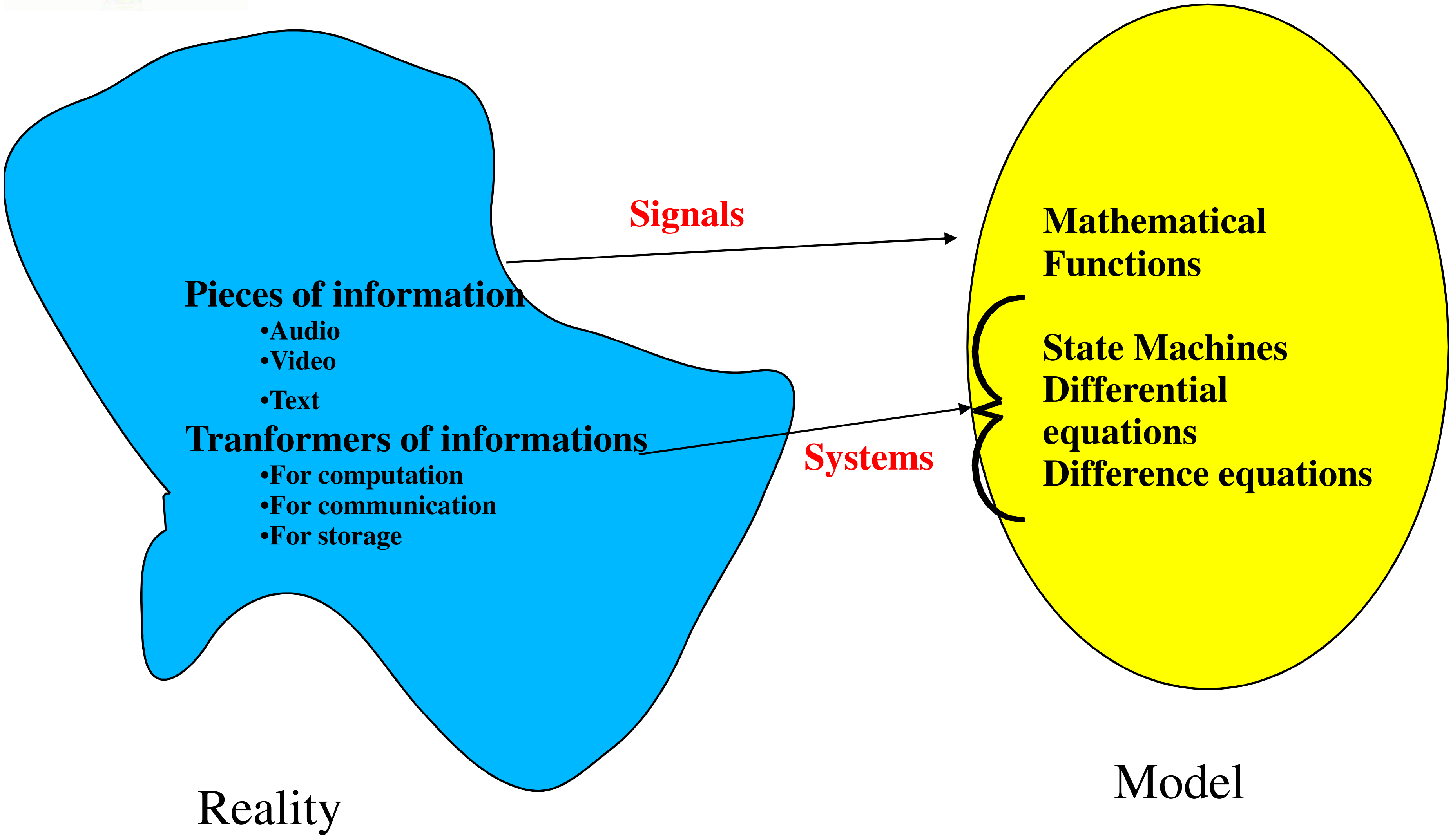
  relevant information on our reality

Modeling reality

- Modeling: act of representing a system or a subsytem formally
- Models are expressed using a modeling language
- A modeling language comprises
  - a set of symbols,
  - a set of rules to combine them (syntax)
  - A set of rules for interpreting combinations of symbols (semantics)

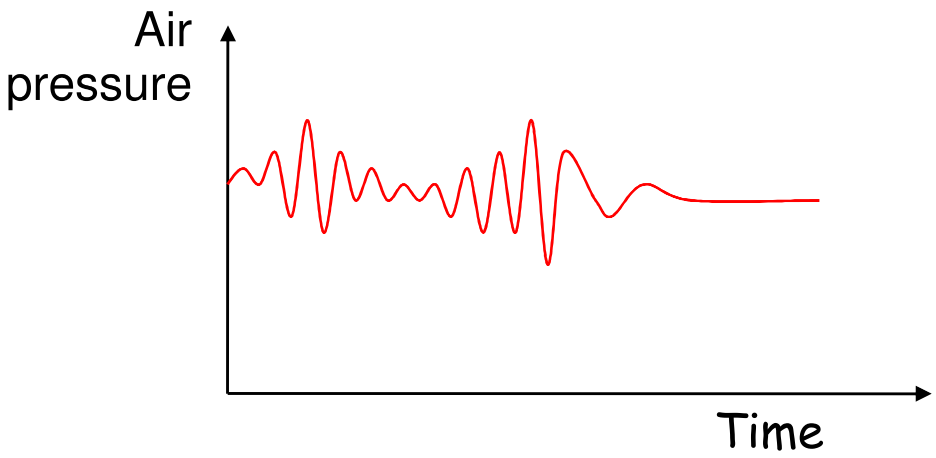


Relevant aspects to model



Example

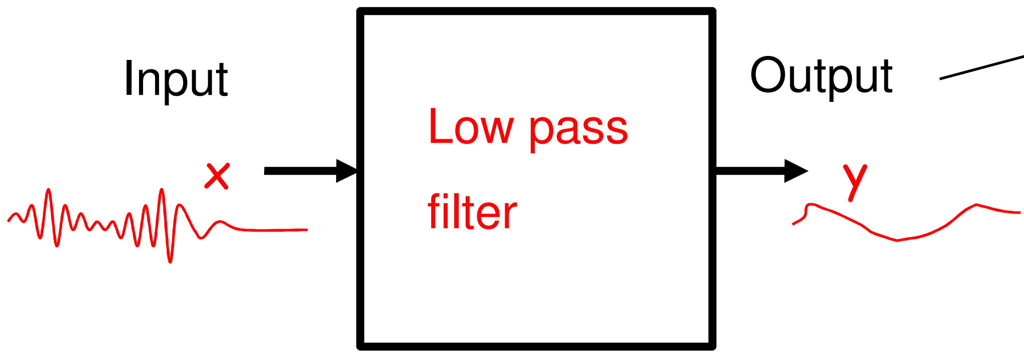
Signal



Function

Sound: Time $\rightarrow$ Air pressure

System



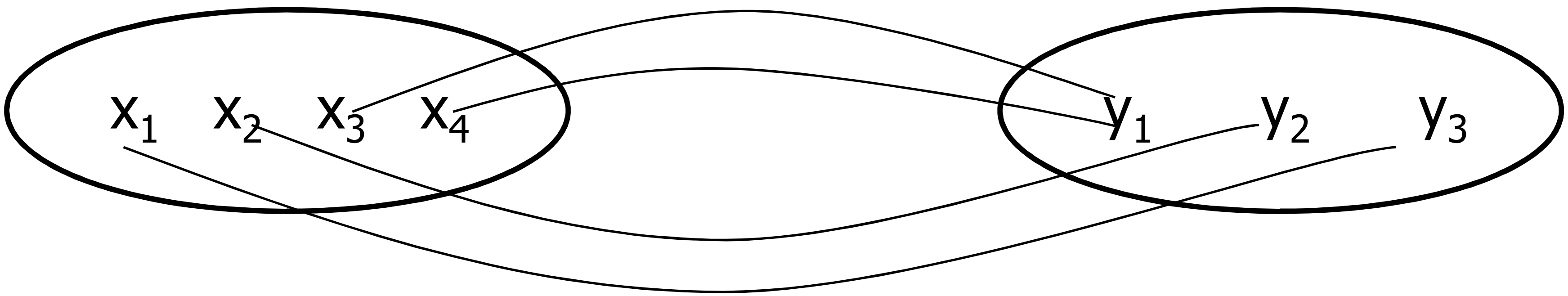
Difference
Equation

Outline

- Mathematical background
  - Signals
  - Systems

Signals

- A signal is a function identified by four things
  - the name (f, g, sin, cos, sound, …)
  - the domain (a set)
  - the range (a set)
  - the graph or assignment (for every domain element, a range element)
- Every element in the domain is associated to one and only one element in the range
- If all elements in the range are paired the function is said ***onto***



Example

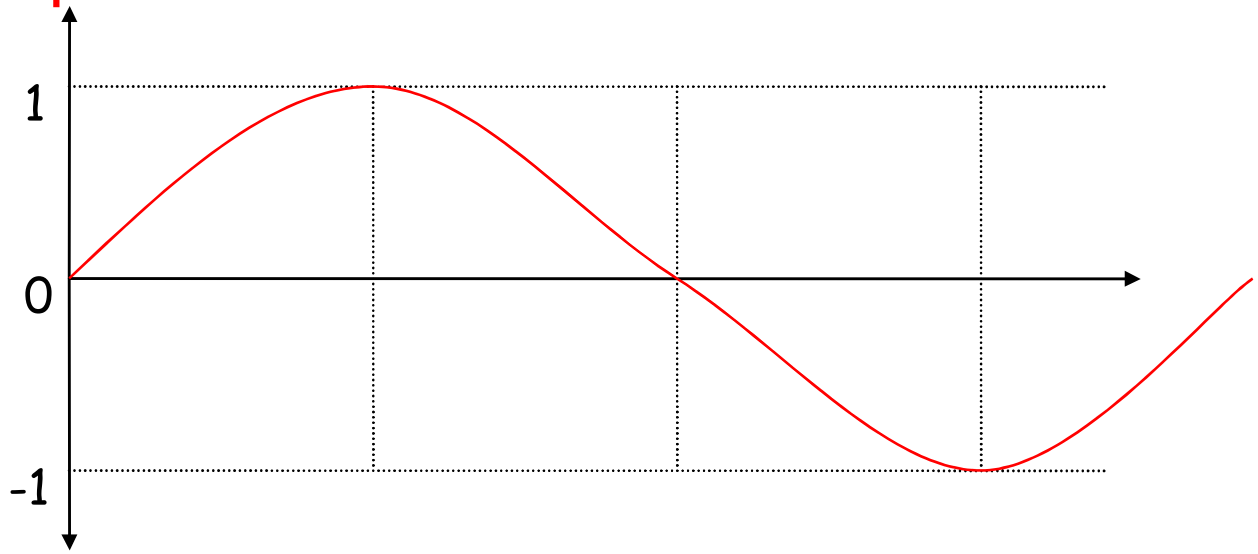
sin, cos

1 Domain : Reals .

2 Range : [-1,1] $= \{ x \in \text{Reals} \mid -1 \leq x \leq 1 \}$.

3 Graph : for each real x, the real sin (x) $\in$ [-1,1]

Graph



Formally, the graph of a function can be thought of as a set of pairs :

$$\{\ (x,y) \in (\ \mathrm{Reals} \times [-1,1]\ )\ \mid\ y = \sin(\ x\ )\ \}$$

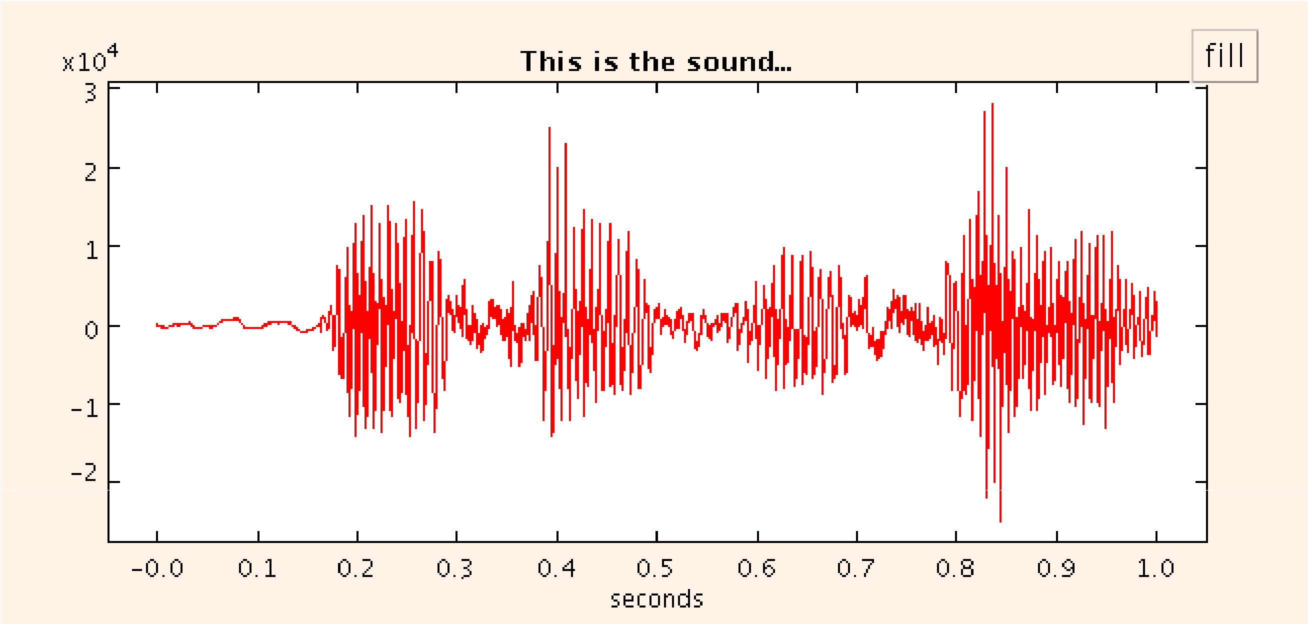
$$= \{\ \dots\ ,(0,0)\ ,\ \dots\ ,(\pi/2,\ 1)\ ,\ \dots\ ,(\pi,0),\ \dots\ ,(3\pi/2,-1),\ \dots\ \}\ .$$

Graph (continued)

If domain and range of a function are finite, then the graph can be given by a table :

x	y	$f(x,y) = x \wedge y$
true	true	true
true	false	false
false	true	false
false	false	false

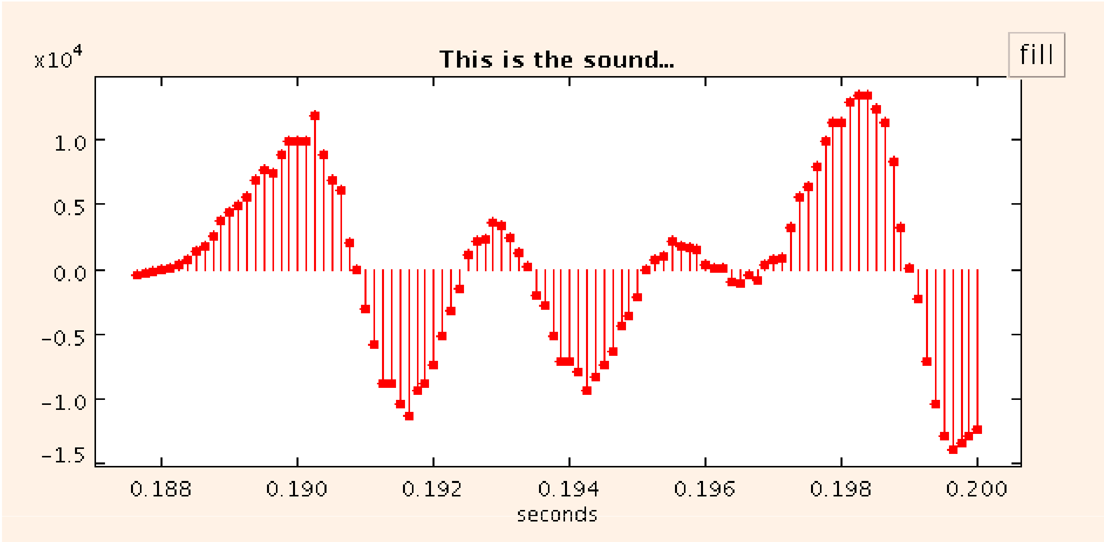
Example Voice



Voice: Time → Pressure
Domain = Time =[0,1]
Range = Pressure = Reals
graph(Voice) = { red points} ∈ Time X Pressure Time

Impossible to handle for a computer

Example Voice - (continued)



8 kHz sampling

A-D conversion

DigitizedVoice: DiscreteTime → Integer16

Domain = DiscreteTime =[0,1/8000, 2/8000]

Range = Integer16

graph(DigitizedVoice) = { red points} ∊ DiscreteTime X Integer16

Example Image

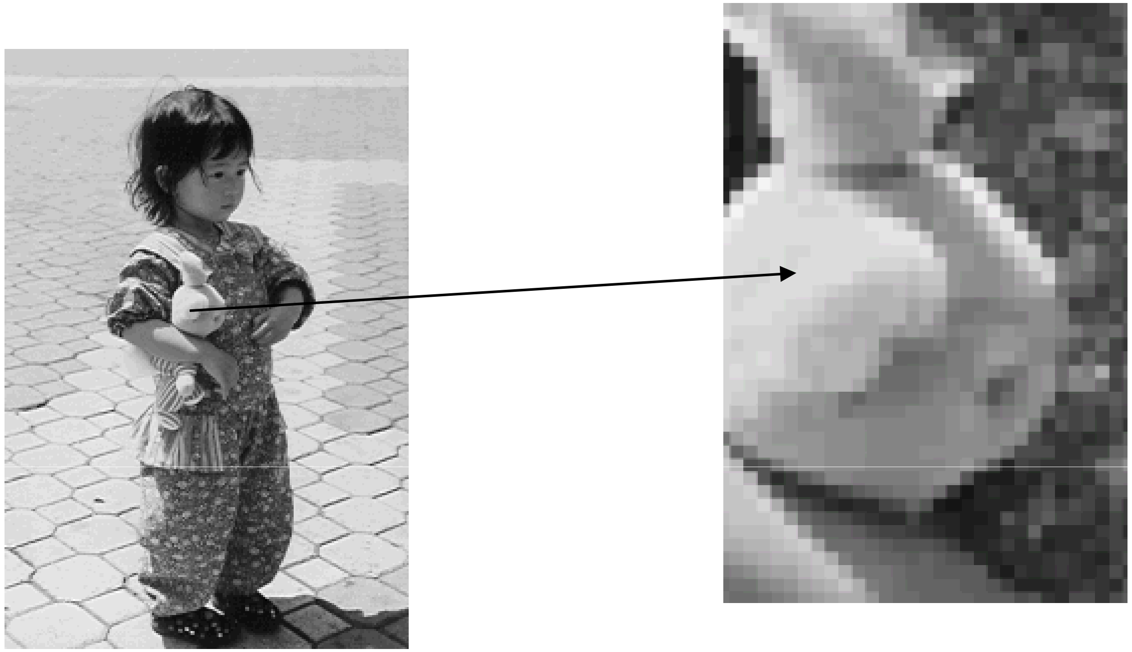
MonochromeImage: VertPos X HorPos→ Intensity

Domain = VertPos X HorPos = [a, b] X [c, d]

Range = Intensity = [IntensityBlack, IntensityWhite]

graph(MonochromeImage = [[x, y], Intensity(x,y)]

Example Digitised Image



Closeup

MonochromeDigitisedImage: DiscreteVertPos X DiscreteHorPos→ DiscreteIntensity

Domain = DiscreteVertPos X DiscreteHorPos = {0,1,....299} X {0,1,2,...,299}

Range = DiscreteIntensity = {0,1,...,256}

graph(MonochromeImage = [[x, y], DiscreteIntensity(x,y)]

Digitised Colour Image

DigitisedColourImage: DisVertPos X DisHorPos→ DisIntensity^3

Domain = DisVertPos X DisHorPos = {0,...,299} X {0,...,199}

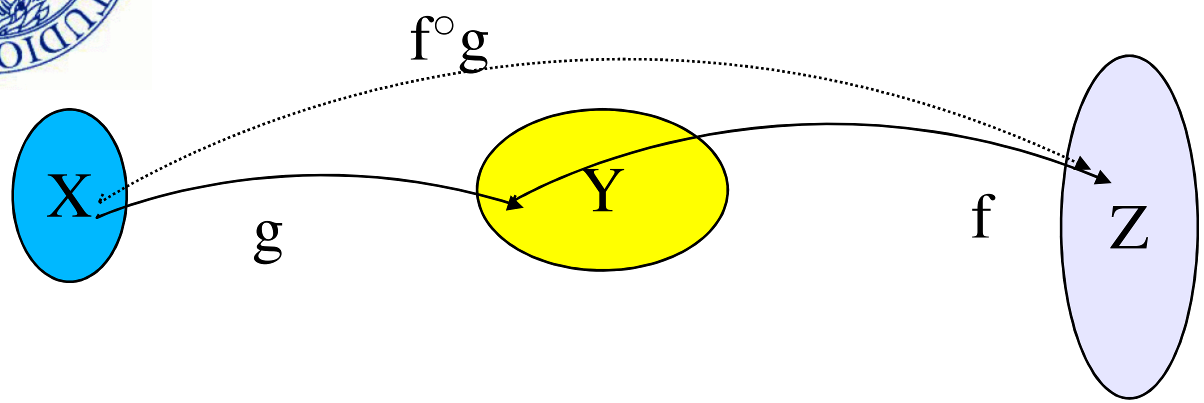
Range = DisIntensity^3 = {0,...,255} X {0, ..., 255} X {0,...,255}

graph(MonochromeImage = [[x, y], [IntensityR(x,y), IntensityG(x,y), IntensityB(x,y)]

Example Sequences

- Sometimes there is no relation whatsoever with Physical time
- Example: Sequence of Events
  - {turn on, start, gear 1, gear 2, gear 1, stop}
- Sequnces are functions whose domain are the natural

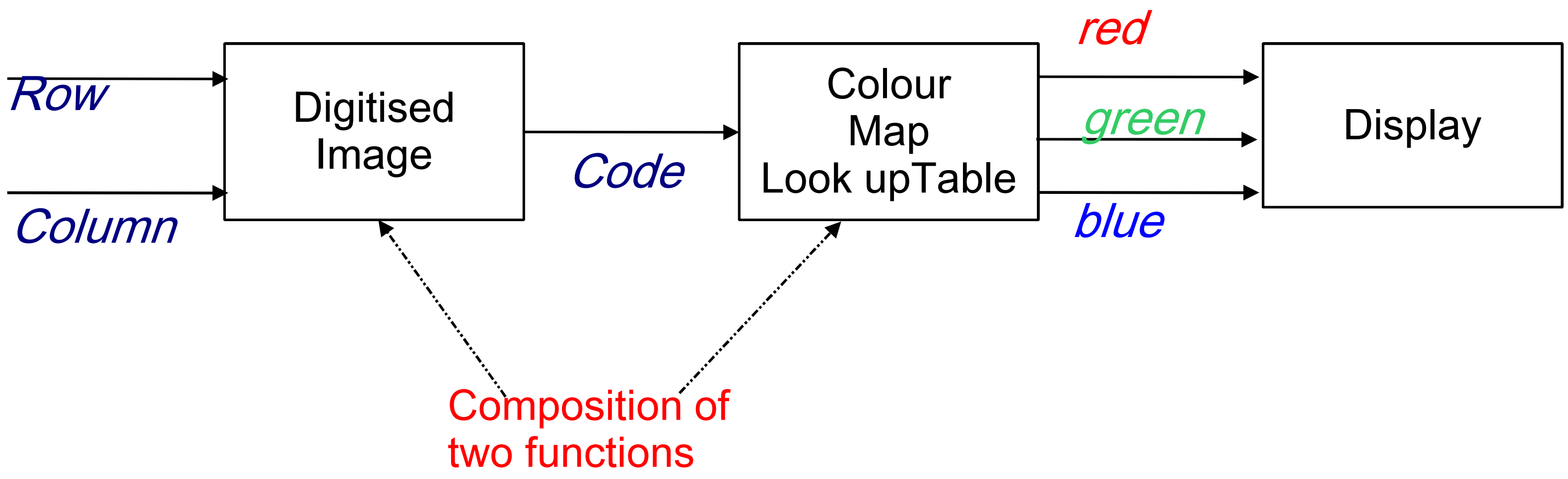
An important concept: function composition



- If $g: X \rightarrow Y$, $f: Y \rightarrow Z$, then $f \times g: X \rightarrow Z$ is defined as follows: $[x,z] \in graph(f^\circ g)$ if and only if there exists $y \in Y$ such that
  - $[x,y] \in graph(g)$
  - $[y,z] \in graph(f)$

Example Compressed Image

- Total colours with RGB coding : 256^3
- It's a lot, and takes a lot of memory!
- Many compression formats use a colour map (e.g., *compuserve GIF*)



Space of signals

- For two sets X, Y
  $[X \to Y] = \{f \mid \text{domain}(f)=X \textit{ and } \text{range}(f)=Y\}$
- Examples:
  - Sounds: [Time → Pressure]
  - MicSounds: [Time → Voltage]
  - BitSequences: [Natural → {0,1}]
  - Images: $[\{0,\ldots,Xres-1\} \times \{0,\ldots,Yres-1\} \to \{0,..,255\}^3]$
  - Movies: [{0,1/30,2/30,...} → Images]

Discrete-time 30 frames/sec

Ways for defining a signal

- Declarative (denotational) definition
  - The signal is identified by expliciting the mathematical relation between the paired elements
  - Example:

$$\forall t \in Time, s(t) = \sin(\omega t)$$

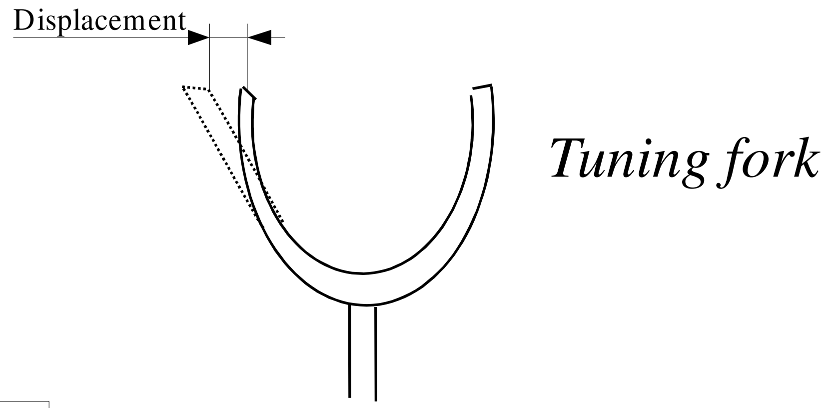
Ways for defining a signal

- Imperative (operational) definition
  - The signal is identified by indicating a way to compute the pairwise association
  - Examples
    - A lookup table
    - Computer Procedure:

```
const double w = ...;
double s(double t) {
        return sin(w*t);
}
```

Procedure invocation
(different from mathematical function)

Physical modeling



Force

Acceleration

Displacement

- Newton's law:

$$f(t)=m\,a(t)=-k\,x(t)$$

- Acceleration's definition:

$$\ddot{x}(t)=a(t)$$

- Differential equation

$$\ddot{x}(t)=-\frac{k}{m}x(t)$$

Tuning fork

- The solution of the differential equation is the family of signals:

$$x(t) = A\cos(\omega t + \phi)$$
$$where\; \omega = \sqrt{\frac{k}{m}}$$

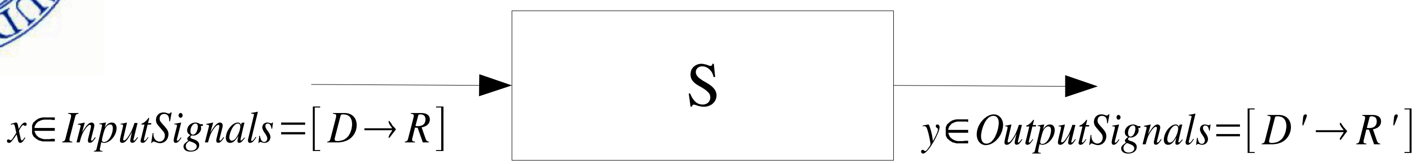
- Values for A and ϕ are determined by the initial conditions

Outline

Mathematical background

- Signals
- Systems

Systems



$$Behaviours(S) = \{(x, y)\ s.t.\ x \in [D \to R] \wedge y = S(x)\}$$

- A system is a function whose domain and range are signal spaces
- A *behaviour* is an input/output pair
- A system could in principle be specified by listing all of its behaviours but this is utmost impractical
- To specify a system we need:
  - a domain (input signals)
  - a range (output signals)
  - the graph

Memoryless systems

- Previous inputs values are forgotten (*do not influence current value*)
- Formally, given a domain X and range Y for signals and a function f: $Y \to Y$, we can define a memoryless system

  S:[$X \times \quad Y$] $\to$ [$X \times Y$]
- by

  $\forall\ x \in X$ and $\forall\ u \in [\ X \times Y]$, $\quad (S(u))(x) = f(u(x))$.

- Example:
  - consider a continuous time system that operates as follows:

  $$\forall\, t \in Reals,\ y(t) = x^2(t)$$

  - according to our definition this is a memoryless system!!

What about this?

$$\forall\, t \in Reals,\ y(t) = \frac{1}{M}\int_{t-M}^{t} x(\tau)\,d\tau = \frac{1}{M}\int_{0}^{M} x(t-\tau)\,d\tau$$

- Is this system memoryless?

- No!!, the integral operator uses the past story of *x(t)*

Continuous-time systems

- Consider a class of systems operating on the class of *ContSignals.*
- *ContSignals* signals could be
- [*Time -> Reals],* or *[Time -> Complex]* where *Time = Real* or *Time = Real*$_+$
- Frequently continuous time systems are defined by means of differential equations

Example

- Motion of a particle
  - the applied force is an input signal

$$f(t) = m\,a(t)$$

$$\forall\, t \in Real \; with \; t \geq 0, \; \ddot{x}(t) = \frac{f(t)}{m}$$

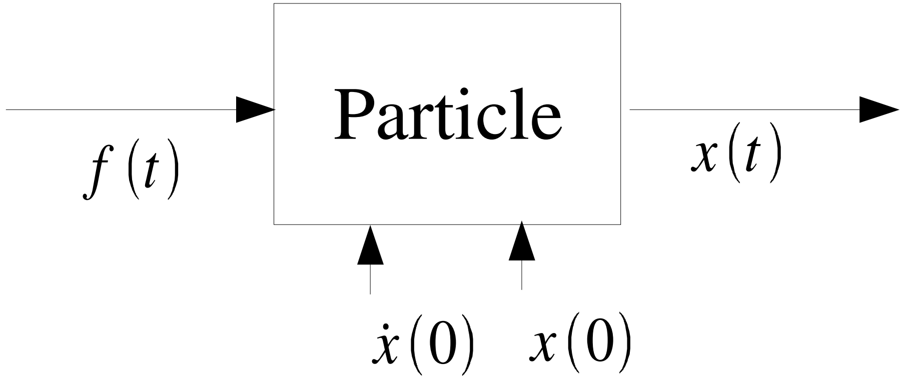
- Integrating twice

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \frac{f(\tau)}{m}\, d\tau$$

$$x(t) = x(0) + \dot{x}(0)\,t + \int_0^t \left[\int_0^s \frac{f(\tau)}{m}\, d\tau \right] ds$$

Example - (continued)

- Block representation



- Example evolution:

$$f(t)=1, x(0)=1, \dot{x}(0)=1 \rightarrow x(t)=1+t+\frac{1}{m}\frac{t^2}{2}$$

$$f(t)=\cos(\omega t), x(0)=0, \dot{x}(0)=0 \rightarrow x(t)=-\frac{1}{m}\frac{\cos(\omega t)-1}{\omega^2}$$

Sinusoidal signal
whose amplitude
decrease with the frequency

Discrete-time equations

- Consider a class of systems operating on the class of Disc*Signals.*
- *DiscSignals* signals could be
- [*Time -> Reals],* or *[Time -> Complex]* where
  Time = Integers, Time =Natural or *Time= Natural$_0$*
- Frequently discrete-time systems are defined by means of difference equations

Example

- Consider a system:
  S:[*Naturals*$_0$ -> *Reals*] -> [*Naturals*$_0$ ->*Reals*] where for all $x \in$[*Naturals*$_0$ -> *Reals*], *S(x) = y* is given by:

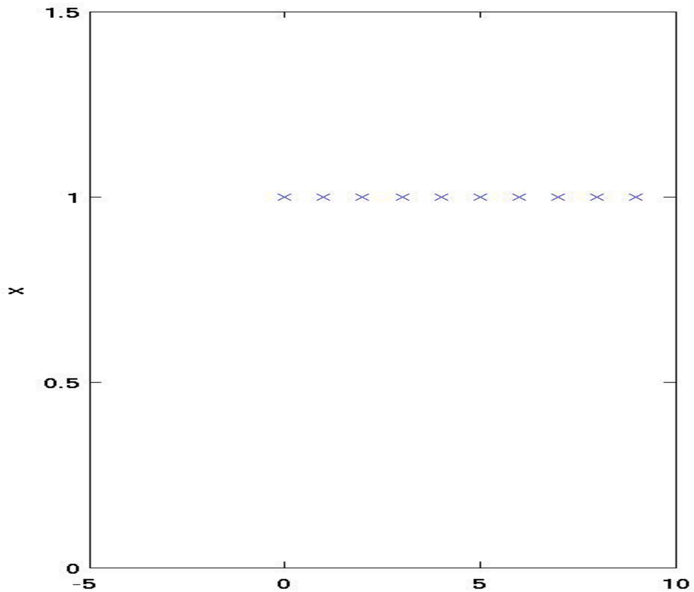
$$\forall n \in Integers\ y(n) = \frac{x(n) + x(n-1)}{2}$$

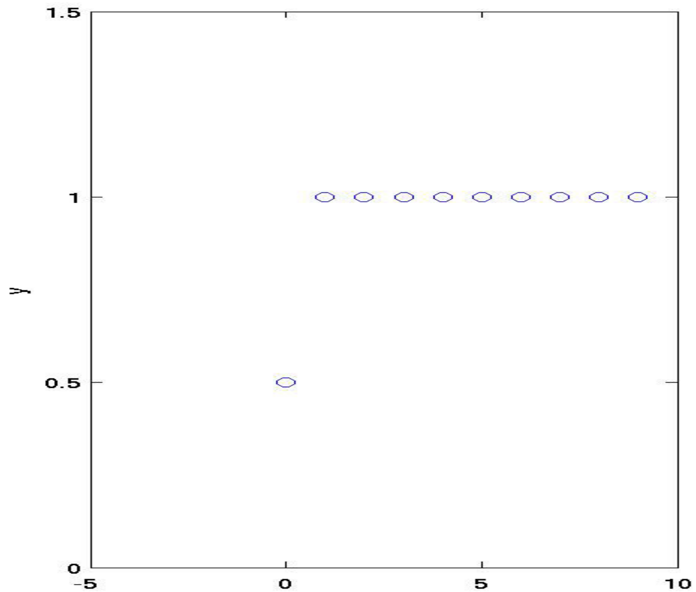
- This operation is a moving average and it can be generalised as follows:

$$\forall n \in Integers\ y(n) = \frac{\sum_{i=0}^{N-1} x(n-i)}{N}$$

Example - (continued)

- Step responses



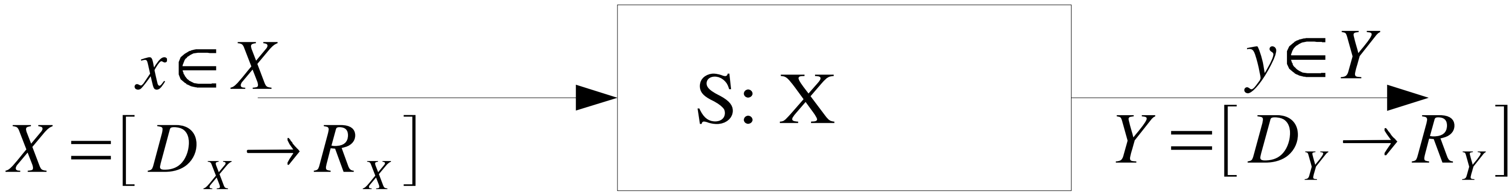


- The effect is smoothing variations (very much used in wall street!)

Block diagrams

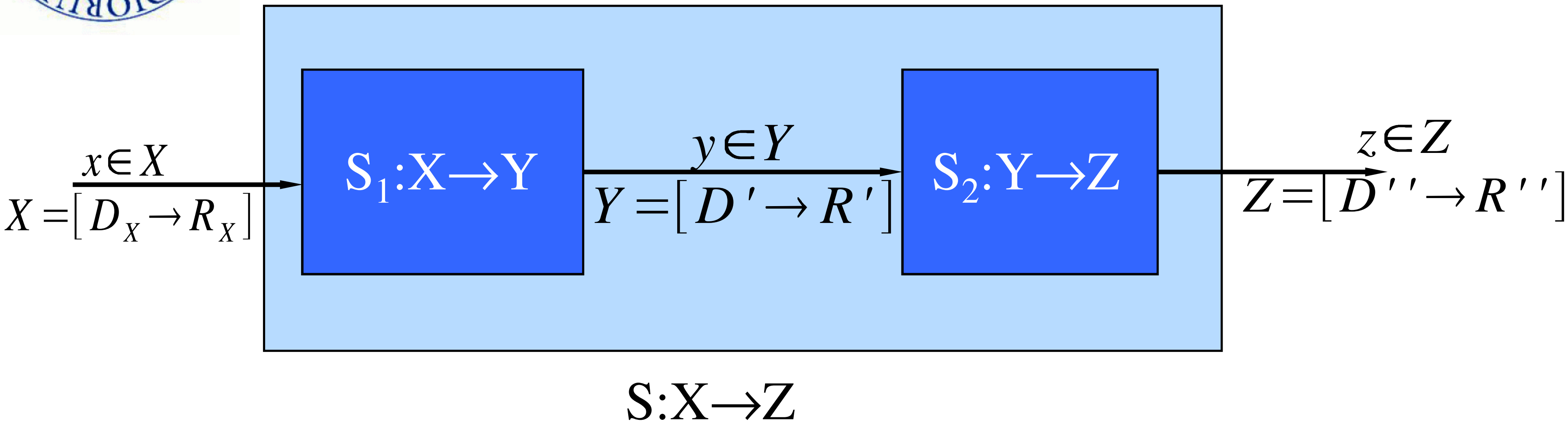
- Thus far we use block diagrams informally
- They can be defined as a full-fledged visual syntax
  - a system is described as an interconnection of other systems, each of which transforms incoming input signals into outgoing output signals
  - Advantages against textual specification are compactness and hierarchical composition

Block diagrams - (continued)



- Blocks are sytems
- Arrows connecting blocks are labeled by signals
- Connecting a system inputs to some other system's outputs amounts to function composition

Cascade connection



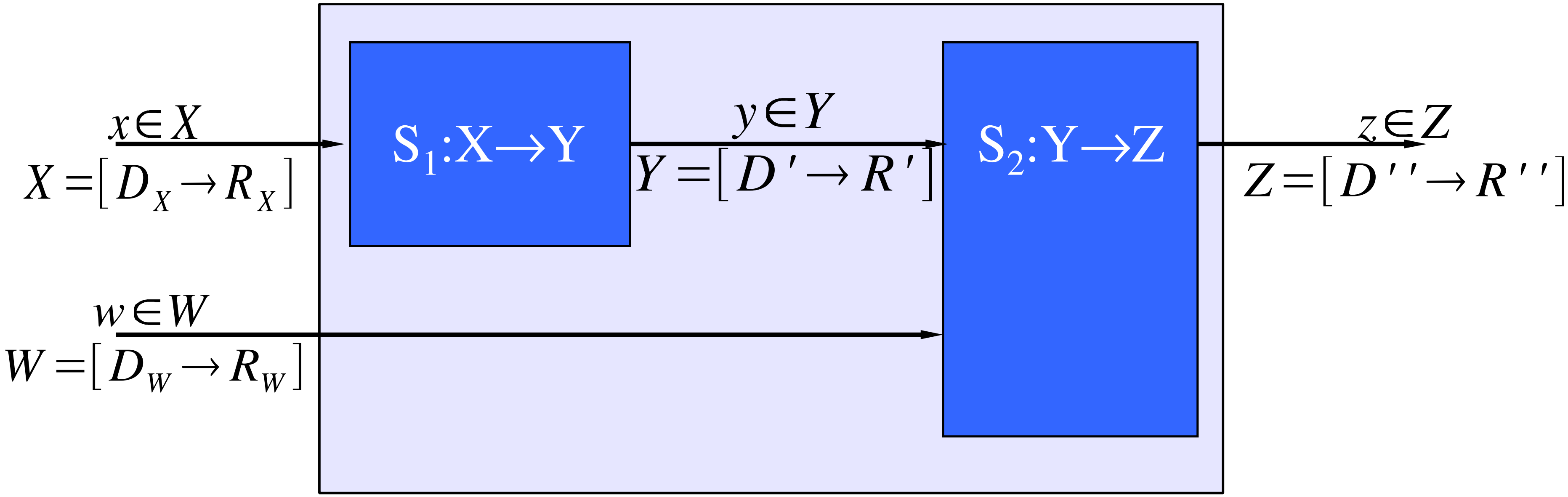
- For the connection to make sense it must hold

$$\text{Range}(S_1) \subseteq \text{Domain}(S_2)$$

- The function is the composition:

$$\forall x \in X,\ S(x) = S_2(S_1(x))$$

A more complicated example

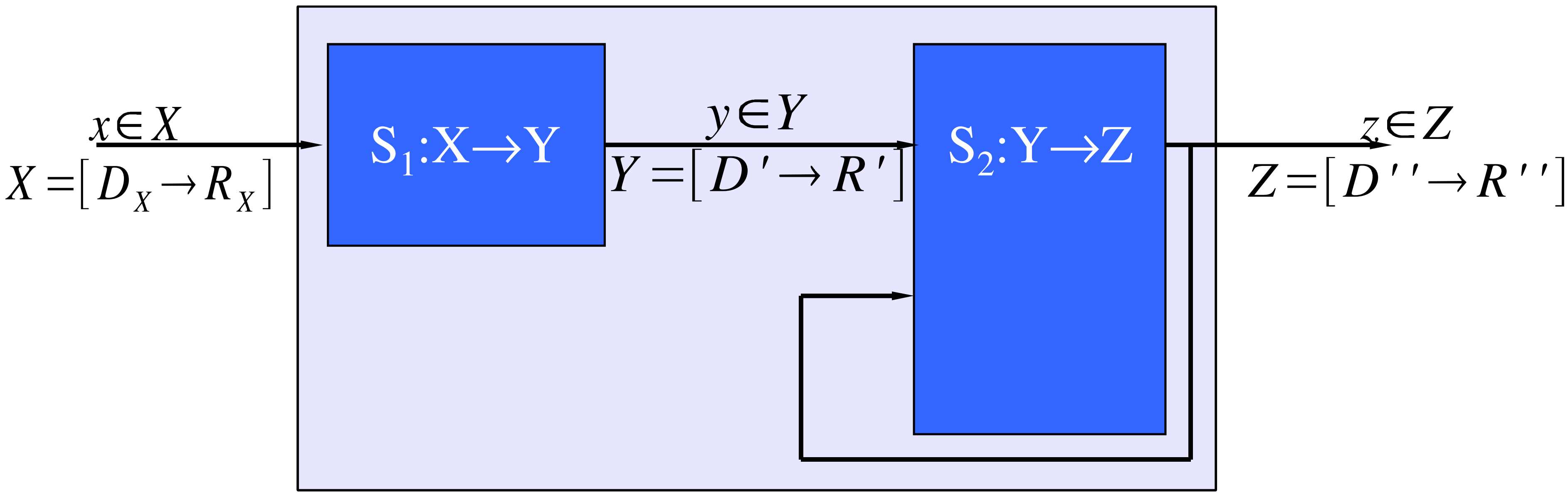


$S' : X \times W \to Z$

- By visual inspection:

$$\forall (x,w) \in X \times W,\ S'(x,w) = S_2(w, S_1(x))$$

A much more complicated example



$$S' \colon X \times W \rightarrow Z$$

- By visual inspection:

S' is at both sides of the expression!!

$$\forall (x) \in X,\ S'(x) = S_2(S'(x), S_1(x))$$

Fixed point

- The definition above is actually an equation

$$\text{set } y = S'(x)$$

$$y = S_2(y, S_1(x))$$

- The solution of the equation *y* is called a fixed point
- Difficulties
  - y is a function
  - the fixed point may not exist
  - if the fixed point exists it might not be unique