Ball and Plate

1. From the stepper motors angles to the plate pitch and roll

The ball and plate assembly can be identified as in figure 1

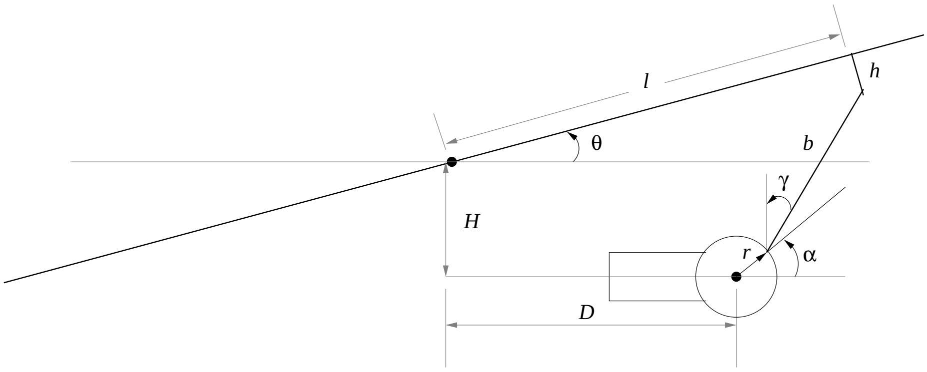

Figure 1. Geometry of the assembly - relationship between α and θ.

l_x	l_y	D_x	D_y	α_{0x}	α_{0y}	H_x	H_y	h	b	r
76	64	66.5	53	0	0	47	49	15	40	12

Table 1. Length of the links in the ball and plate assembly

The relationship between α and θ is given by the closed chain geometry (neglecting the elastic coefficient of the plate structure).

$$H + l\sin\theta = r\sin\alpha + b\cos\gamma + h\cos\theta$$
$$l\cos\theta + h\sin\theta = D + r\cos\alpha + b\sin\gamma$$

γ can be eliminated yielding.

$$\begin{aligned}(H + l\sin\theta - r\sin\alpha - h\cos\theta)^2 \;+ \\ (l\cos\theta + h\sin\theta - D - r\cos\alpha)^2 \;=\; b^2\end{aligned}$$

At equilibrium, it must be $\theta = 0$. Therefore,

$$(H - r\sin\alpha_0 - h)^2 + (l - D - r\cos\alpha_0)^2 = b^2$$

For the x coordinate, it is

$$(32 - 12\sin\alpha_{x0})^2 + (9.5 - 12\cos\alpha_{x0})^2 = 40^2$$

which can be solved, yielding $\alpha_{x0} \approx -41.5°$.
For the y coordinate, it is

$$(34 - 12\sin\alpha_{y0})^2 + (11 - 12\cos\alpha_{y0})^2 = 40^2$$

that is, $\alpha_{y0} \approx -30°$.
going back to our $\theta = f(\alpha)$,

$$\begin{aligned} H^2 + D^2 + r^2 + h^2 + l^2 +& \\ 2\sin\theta(lH - rl\sin\alpha - hD - rh\cos\alpha) +& \\ 2\cos\theta(rh\sin\alpha - Hh - lD - rl\cos\alpha) +& \\ -2rH\sin\alpha + 2rD\cos\alpha = b^2& \end{aligned}$$

linearizing around $(\theta, \alpha) = (0, \alpha_0)$ we have

$$\begin{aligned} H^2 + D^2 + r^2 + h^2 + l^2 +& \\ 2\theta(lH - rl(\alpha\cos\alpha_0 + \sin\alpha_0) - hD - rh(\cos\alpha_0 - \alpha\sin\alpha_0)) +& \\ 2(rh(\alpha\cos\alpha_0 + \sin\alpha_0) - Hh - lD - rl(\cos\alpha_0 - \alpha\sin\alpha_0)) +& \\ -2rH(\alpha\cos\alpha_0 + \sin\alpha_0) + 2rD(\cos\alpha_0 - \alpha\sin\alpha_0) = b^2& \end{aligned}$$

$$\theta = \frac{H^2 + D^2 + r^2 + h^2 + l^2 - b^2 + 2(r(h - H)(\alpha\cos\alpha_0 + \sin\alpha_0) - Hh - lD + r(D - l)(\cos\alpha_0 - \alpha\sin\alpha_0))}{2(rl(\alpha\cos\alpha_0 + \sin\alpha_0) + hD + rh(\cos\alpha_0 - \alpha\sin\alpha_0) - lH)}$$

$$\theta_x = \frac{11176,25 + 2(-384(\alpha\cos\alpha_0 + \sin\alpha_0) - 5759 - 114(\cos\alpha_0 - \alpha\sin\alpha_0))}{2(912(\alpha\cos\alpha_0 + \sin\alpha_0) + 180(\cos\alpha_0 - \alpha\sin\alpha_0) - 2574,5)}$$

Finally, substituting the α_0 values we have

$$\theta_x = \frac{-363.1\alpha}{802.33\alpha - 3044} \qquad \theta_y = \frac{-419.34\alpha}{755.11\alpha - 2569.1}$$

2. From the plate pitch and roll to the ball acceleration (and position)

The dynamics of the ball motion on the non-inertial plate system can be analyzed with reference to figure 2.

$R = mgcos(\theta)$	reaction normal to the plate
$F_a = \mu_s R$	(static) friction force
$mgsin(\theta) - F_a = ma$	translation on the plate axis
$F_a r = I\alpha$	rotation with respect to the center of the sphere
$\alpha r = a$	boll rotates without slipping

the momentum of inertia for a sphere is $I = 2/5mr^2$. By substituting in the previous, it is

$$\begin{aligned} mgsin(\theta) - \mu_s mgcos(\theta) &= ma \\ \mu_s mgcos(\theta) &= 2/5ma \end{aligned}$$

and the (well-known)

$$a = 5/7gsin(\theta)$$

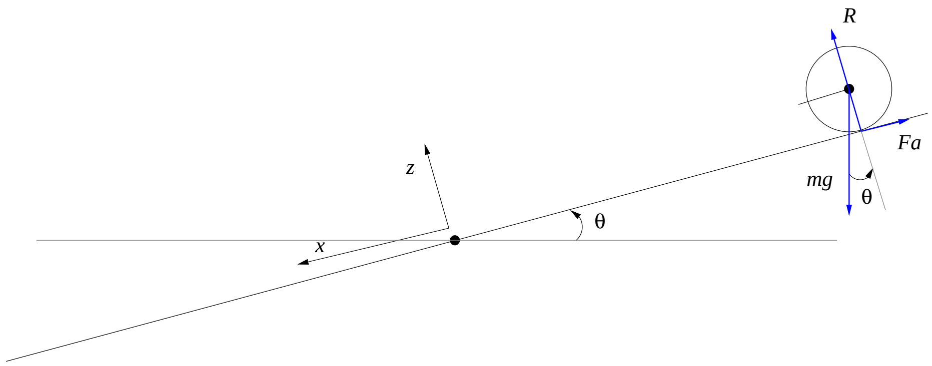

Figure 2. Ball dynamics on a plate.